Morse Theory & Floer Homology

Lecture * ?

The Topology of Sublevel Sets

We know from the previous lecture that the topology of the level set of a Morse function won't change untill we meet a critical point, the same thing will happen on the sublevel set:

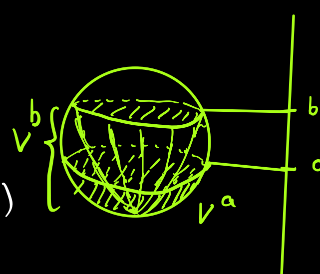
$$V^a = f^{-1}((-\infty, a]).$$

<u>Theorem 1</u>: f is the Morse function defined on V, there are no critical values of f in the interval $[a, b]$, and $f^{-1}([a,b])$ is compact, then V^b is diffeomorphic to V^a.

<u>Proof</u>: We're going to construct a flow of a vector field ψ^s, such that:

$$\psi^s : V^b \xrightarrow{\simeq} V^a.$$

To see this, we need to let ψ^s satisfies:



$$(\psi^s)^{-1} f^{-1}((-\infty, a]) = f^{-1}((-\infty, b])$$

$$\Leftrightarrow \quad (f \circ \psi^s)^{-1}((-\infty, a]) = f^{-1}((-\infty, b]).$$

To construct such a flow, we assume X is the pseudo gradient of f. Since there're no critical values in $[a, b]$, it allows us to define a function:

$$\rho(x) = \begin{cases} -\dfrac{1}{(df)_x X(x)} & x \in f^{-1}([a,b]) \\ 0 & x \notin f^{-1}([a,b]) \end{cases}$$

We let ψ^s be the flow induced by $Y = \rho X$, then by computing:

$$\frac{d f\circ\psi^s_{(x_0)}}{ds} = (df)_{\psi^s_{(x_0)}} \frac{d\psi^s(x_0)}{ds}$$

$$= (df)_{\psi^s_{(x_0)}} Y(\psi^s_{(x_0)})$$

Here we may assume $\psi^s_{(x_0)} \in f^{-1}([a,b])$

$$= (df)_{\psi^s_{(x_0)}} \left(- \frac{X(\psi^s_{(x_0)})}{(df)_{\psi^s_{(x_0)}} X(\psi^s_{(x_0)})} \right)$$

$$= -1$$

thus $f\circ\psi^s(x) = -s + c(x)$, by letting $s=0$, $c(x) = f(x)$, i.e:

$$f\circ\psi^s_{(x)} = -s + f(x), \quad x \in f^{-1}([a,b])$$

now, if $f\circ\psi^s(x) \le b$, we have $-s + f(x) \le b \Rightarrow f(x) \le s + b$

thus $\psi^{b-a} : V^b \xrightarrow{\cong} V^a$ is diffeomorphism.

<u>Remark</u>: The family

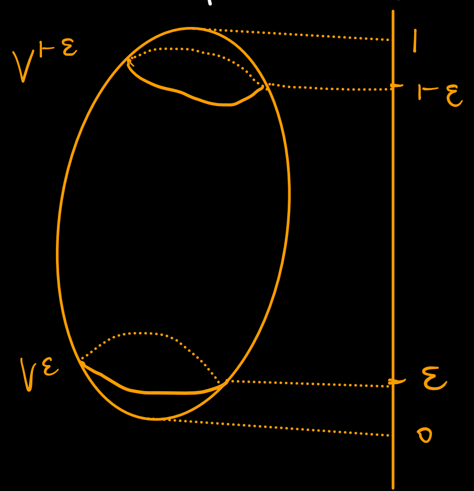
$$r : V^b \times I \longrightarrow V^a$$

$$(x,t) \longmapsto \begin{cases} x & f(x) \le a \\ \psi^{t(f(x)-a)}(x) & a \le f(x) \le b \end{cases}$$

is a deformation retraction from V^b to V^a, actually $r_0 = id_{V^b}$, r_t is an inclusion on V^a, r_1 has image V^a, thus V^a is a deformation of V^b.
Later, we'll show that, if the interval $[a,b]$ cross a critical value p, things will change but not a big deal, at least in a small nbhd of p, namely $[p-\varepsilon, p+\varepsilon]$, $V^{p+\varepsilon}$ can deformate to $V^{p-\varepsilon}$ attached with an $\mathrm{ind}(p)$-cell.

Now, we consider an extremely case, Morse function of V contains only 2 critical points, WLOG namely, $0, 1$, then $\exists\, \varepsilon > 0$, s.t. $f^{-1}([0,\varepsilon])$ and $f^{-1}([1-\varepsilon, 1])$ are disk $\mathbb{D}^n$, then Applying the Thm on $[\varepsilon, 1-\varepsilon]$, we know that $V^\varepsilon \cong V^{1-\varepsilon} \cong \mathbb{D}^n$, thus V is made by the gluing of the 2 disks $V^{1-\varepsilon}$ and $f^{-1}([1-\varepsilon,1])$, that is V is a sphere.

Thm 2 (Reeb's Theorem) If V is compact and allows a morse function only with 2 critical points, then V is homeomorphic to a sphere.



$$\begin{tikzcd} f^{-1}([1-\varepsilon,1]) \arrow[r, "\varphi_1"] & D^n \\ V^{1-\varepsilon} \arrow[u, "\varphi_2\circ\varphi_3\circ\varphi_1^{-1}"] \arrow[r, "\varphi_2"] & V^{\varepsilon} \arrow[u, "\varphi_3"'] \end{tikzcd}$$

$$\Downarrow$$

$$\partial f^{-1}([1-\varepsilon,1]) \xrightarrow{\varphi} \partial V^{1-\varepsilon}.$$

Proof: We just need to give a explicit expression of that homeomorphism.

Recall that, the standard sphere S^{n-1} is made by attaching two n-cells by identifying the boundary with the identity mapping:

$$S^{n-1} = D_1^n \amalg D_2^n / \mathrm{Id}.$$

The V in we here is by attaching two n-cells by identifying the boundary with the φ:

$$V = D_1^n \amalg D_2^n / \varphi.$$

By letting $h: S^{n-1} \longrightarrow V$,

$$h(x) = \begin{cases} x & x \in D_1^n \\ \|x\|\,\varphi\left(\frac{x}{\|x\|}\right) & x \in D_2^n - \{0\} \\ 0 & x = 0. \end{cases}$$

Next, we shall consider the case of crossing a critical value.

Remark 1: The resut also holds for a degenerate critical point, but the proof is complicated.

2. We only prove the "Homeomorphic", but not a diffeomorphic, this is the idea of the construction of the Milnor's exotic sphere S^7.

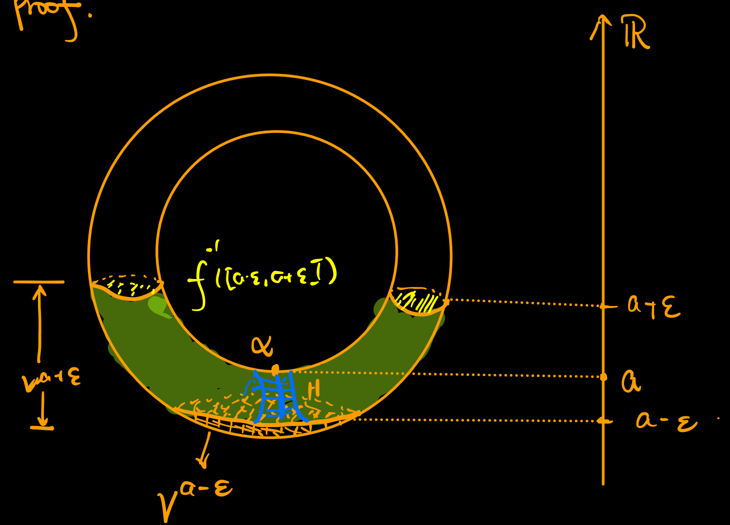
Thm 3: Let $f: V \longrightarrow \mathbb{R}$ be a Morse function. α is critical point with index k, we assume that, $(f(\alpha) = a)$

i). For $\forall \varepsilon > 0$, f has no critical points in $[a-\varepsilon, a+\varepsilon]$,

ii). $f^{-1}([a-\varepsilon, a+\varepsilon])$ is compact,

Then the homotopy type of $V^{a+\varepsilon}$ is $V^{a-\varepsilon}$ attaching a k-cell.

The idea of the proof.



We will modify f to a new function F, such that F only differs to f from a small nbghd H near the critical point α, then $F^{-1}((-\infty, a+\varepsilon]) = V^{a+\varepsilon}$, and $F^{-1}((-\infty, a-\varepsilon]) \simeq V^{a-\varepsilon} \cup H$, we'll make effort to show that H can deformate to a k-cell e^k, then by applying thm 1, that will ends the proof.

Proof: Choose a Morse chart (U, u^i) at p, then:

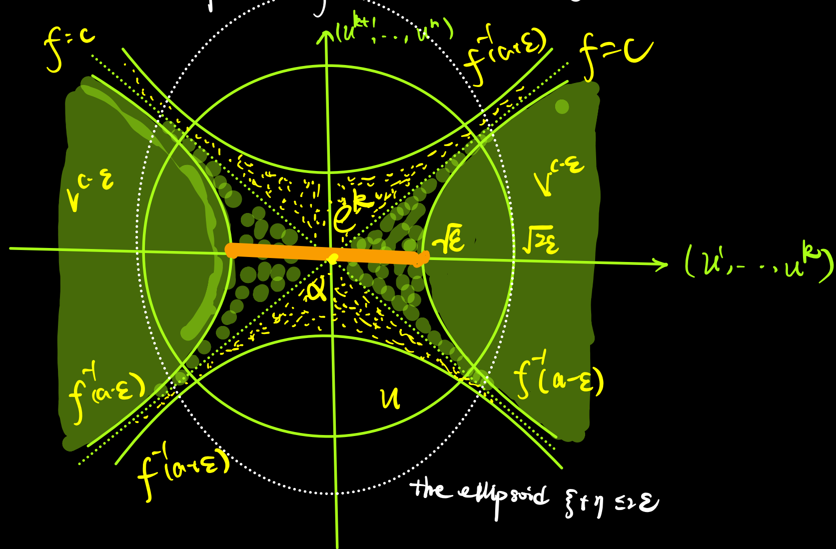
$$f(u^1, \dots, u^n) = a - (u^1)^2 - \dots - (u^k)^2 + (u^{k+1})^2 + \dots + (u^n)^2.$$

the coordinate of α is actually $(0, \dots, 0)$.

We may assume $f^{-1}([a-\varepsilon, a+\varepsilon])$ is compact and has no critical points except α, and $\mathcal{U}$ contains a $\sqrt{2\varepsilon}$-ball, define e^k be the set:

$$e^k := \{p \in \mathcal{U} : (u^1)^2 + \cdots + (u^k)^2 \le \varepsilon,\ u^{k+1} = \cdots = u^n = 0\}$$

where u^i in the $\{\}$ represents for the coordinate of p in $\mathcal{U}$, as it is a k-cell,



Note that $V^{c-\varepsilon} \cap e^k = \partial e^k$, so e^k is attached to $V^{c-\varepsilon}$ as required.

Now we construct F as follows: define: $\mu : \mathbb{R} \longrightarrow \mathbb{R}$, with

$$\begin{cases} \mu(0) > \varepsilon & \\ \mu(r) = 0, & r \ge 2\varepsilon \\ -1 < \mu'(r) \le 0, & \forall r \in \mathbb{R} \end{cases}$$

Let

$$F(x) = \begin{cases} f(x) & x \in V \setminus \mathcal{U} \\ f(x) - \mu\left((u^1)^2 + \cdots + (u^k)^2 + 2(u^{k+1})^2 + \cdots + 2(u^n)^2\right), & x \in \mathcal{U}, \end{cases}$$

where $(u^1, \ldots, u^n)$ represents the local coordinate of x in $\mathcal{U}$-chart, then F is a smooth function through entire V.

(To be convienient, we denote by $\xi, \eta : \mathcal{U} \longrightarrow \mathbb{R}$, instead of the function:

$$\xi = (u^1)^2 + \cdots + (u^k)^2$$
$$\eta = (u^{k+1})^2 + \cdots + (u^n)^2$$

then in u-chart, $f = a - \xi + \eta$, $F = f - \mu(\xi + 2\eta) = c - \xi + \eta - \mu(\xi + 2\eta)$).

<u>Claim 1</u>: $F^{-1}((-\infty, a+\varepsilon]) = V^{c+\varepsilon}$.

In fact, outside the ellipsoid $\xi + \eta \le 2\varepsilon$, $F = f$, inside the region, we have:

$$F \le f = a - \xi + \eta \le a + \frac{1}{2}\xi + \eta \le a + \varepsilon$$

<u>claim 2</u>: F has the same critical points as f.

$$\begin{cases} \dfrac{\partial F}{\partial \xi} = -1 - \mu'(\xi + 2\eta) < 0 \\[2ex] \dfrac{\partial F}{\partial \eta} = 1 - 2\mu'(\xi + 2\eta) \geq 1 \end{cases} \qquad dF = \frac{\partial F}{\partial \xi} d\xi + \frac{\partial F}{\partial \eta} d\eta$$

thus $dF = 0$ iff $d\xi = d\eta = 0$, this only happens at the origion, thus F only has critical points at the origin in U (that is the critical point α of f). F coincides with f outside U.

Now, since $F^{-1}([a-\varepsilon, c+\varepsilon]) \subset f^{-1}([a-\varepsilon, a+\varepsilon])$, thus it is compact, and contains no critical points except α, however, $F(\alpha) = a - \mu(0) < a - \varepsilon$, thus no critical values in $[a-\varepsilon, c+\varepsilon]$!

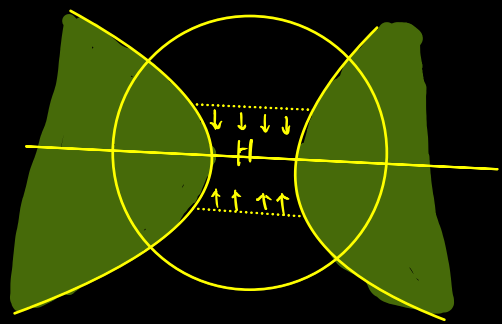
$$\Rightarrow V^{a+\varepsilon} \cong F^{-1}((-\infty, a-\varepsilon])$$

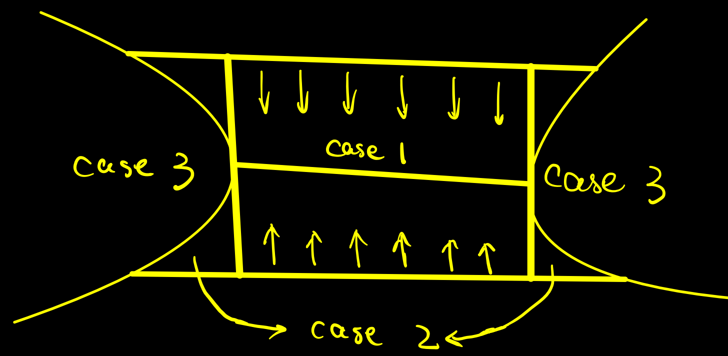
and $F^{-1}((-\infty, a-\varepsilon])$ is a deformation retraction of $V^{a+\varepsilon}$, now, denote:

$$F^{-1}((-\infty, a-\varepsilon]) := V^{a-\varepsilon} \cup H$$

where $H = \overline{F^{-1}((-\infty, a-\varepsilon]) \setminus V^{a-\varepsilon}}$, which contains α, and e^k, in fact, $\frac{\partial F}{\partial \xi} < 0$, the points in e^k indicates that $\xi \le \varepsilon$, $\eta = 0$, thus:

$$F \le F(0) < a - \varepsilon.$$
$$f = c - \xi \geq c - \varepsilon$$





<u>Claim 3</u>: $V^{a-\varepsilon}\cup e^k$ is the deformation retraction of $V^{a-\varepsilon}\cup H$, thus $V^{a-\varepsilon}\cup e^k$ is the deformation retraction of $V^{a+\varepsilon}$.

In case 1, let $r_t: I\times H\longrightarrow e^k$, $(u^1,\dots,u^n)\longmapsto(u^1,\dots,u^k,tu^{k+1},\dots,tu^n)$

$r_1=id_H$, r_0 is the inclusion into e^k.

In case 2, in region $\varepsilon\le\xi\le\eta+\varepsilon$, we define:

$$r_t:(u^1,\dots,u^n)\longmapsto\left(u^1,\dots,u^k,\left(t+(1-t)\sqrt{\frac{\xi-\varepsilon}{\eta}}\right)u^{k+1},\dots,\left(t-(1+t)\sqrt{\frac{\xi-\varepsilon}{\eta}}\right)u^n\right)$$

then r_0 maps the entire region into the hypersurface $f^{-1}(a-\varepsilon)$, $r_1=id$, and if $\xi=\varepsilon$, this coincides with case 1.

In case 3 it suffices to let the r_t be identity map.

This ends the proof.

This Theorem actually gives us a method to decompose a manifold into cellular complexes.

<u>Theorem 4</u>: For a manifold M, let f be a Morse function on M, if a is a critical point of f with index k, and $\exists\,\varepsilon>0$, st $f^{-1}([f(a)-\varepsilon, f(a)+\varepsilon])$ is compact and has no other critical points, then M consists of a k-cell.

A very simple example is the $\mathbb{CP}^n$, we can define a Morse function via the homogenous coordinate:

$$f[z_0, \cdots, z_n] = \frac{\sum_{i=0}^{n} i|z_i|^2}{\sum_{i=0}^{n} |z_i|^2} : \mathbb{CP}^n \longrightarrow \mathbb{R}$$

then, it has $n+1$ distinct critical points, $[1, \cdots, 0]$, $[0, 1, \cdots, 0]$, $\cdots$, $[0, \cdots, 0, 1]$, with index $0, 2, \cdots, 2n$, thus $\mathbb{CP}^n$ has a 0-cell, a 2-cell, $\cdots$, a $2n$-cell.