A Brief Introduction to a Corollary of Schwarz's Lemma

ABSTRACT

In this paper, we are aiming to introduce one of Lars V.Ahlfors' work, Theorem A in [1], which is an extension of Schwarz's lemma,and claims that any holomorphic map from a Poincaé Disk to a Riemann Surface with negative curvature is a Lipschitz map, with Lipschitz constant 1.

Notations Review

It is a quite useful technique in complex analysis which is known as the Schwarz's Lemma:

Lemma 1 (Schwarz's Lemma)**.** *Let $\mathbb{D}$ be a unit disk in $\mathbb{C}$, $f : \mathbb{D} \longrightarrow \mathbb{D}$ is a holomorphic map with $f(0) = 0$, then*
(1).$|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ and $|f'(0)| \leq 1$
(2).If there is another z_0 differs from 0 such that $|f(z_0)| = |z_0|$ or $|f'(0)| = 1$, then f is a rotation.

The proof is based on an application of the maximal module principle, and it has widely application such as deduce the elements in $\mathrm{Aut}(\mathbb{D})$ and $\mathrm{Aut}(\mathbb{H})$ [2].

The next notion is about the hyperbolic geometry [3]. Consider the pseudo Riemannian metric on $\mathbb{R}^3$ given by

$$\langle x, x \rangle = -x_0^2 + x_1^2 + x_3^2$$

It is a Riemannian metric on the hyperbolic space $H^2 = \{x \in \mathbb{R}^3 : \langle x, x \rangle = -1, x_0 > 0\}$.Choose $s = (-1, 0, 0)$, define the pseudo inversion with pole s by

$$f(x) = s - \frac{2(x - s)}{\langle x - s, x - s \rangle}$$

We gain the homeomorphism from the hyperbolic space H^2 to the unit disk $\mathbb{D}$, that induces a Riemannian metric g on the unit disk, which is called the Poincaré disk $(\mathbb{D}, g)$:

$$g = \frac{4(\mathrm{d}x^2 + \mathrm{d}y^2)}{(1 - x^2 - y^2)^2}$$

It is isometric to the hyperbolic space thus has the negative constant (sectional) curvature $K = -1$ [4].

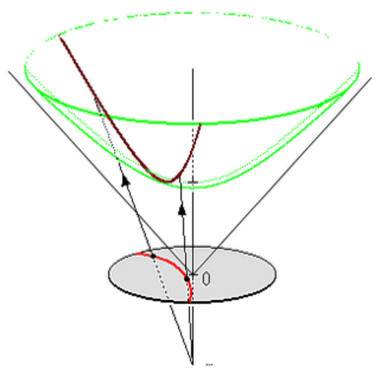

Figure 1: pseudo inversion.

Next we shall consider a Riemann Surface M equipped with a Riemannian metric $\mathrm{d}s^2$:[1]

$$\mathrm{d}s^2 = \lambda^2 \mathrm{d}w \otimes \mathrm{d}\overline{w}$$

Which is expressed in the local coordinate and λ is a holomorphic function on the Riemann surface M.Well, in Ahlfors' paper [1] who allowed λ to be 0, although those points are the singularities of the metric.

It is well known that the Gaussian curvature of that metric is

$$K = -\lambda^{-2} \triangle \log \lambda$$

where the symbol $\triangle$ denotes for the Laplacian operator $\triangle = \partial_x^2 + \partial_y^2 = \partial\overline{\partial}/4$.

[1] In Ahlfors' original paper [1],he used the notion $\mathrm{d}s = \lambda|\mathrm{d}w|$, it is equivalent to what we used here, and it is actually the Hermitian metric on 1-complex manifold.

Ahlfors' Work

Now we are interested in the case of a metric such that the Riemann Surface M has negative curvature, bounded away from zero. It is convenient to choose the upper bound of the curvature equal to -4[2], thus we have

$$\triangle \log \lambda \geq 4\lambda^2$$

If we set $u = \log \lambda$ this is equivalent to

$$\triangle u \geq 4e^{2u}$$

Now we choose the hyperbolic metric on $\mathbb{D}$ via

$$\mathrm{d}\sigma^2 = \frac{\mathrm{d}z \otimes \mathrm{d}\bar{z}}{(1-|z|^2)^2}$$

such that the Poincaré disk has constant sectional curvature $k = -4$.

Now consider $f : (\mathbb{D}, \mathrm{d}\sigma) \longrightarrow (M, \mathrm{d}s)$ the holomorphic map between two Riemann surfaces, so now we have the relation between there line elements (the metric):

$$\mathrm{d}s = \lambda|\mathrm{d}w| = \lambda|\tilde{f}'(z)||\mathrm{d}z|$$

Where $\tilde{f}$ denotes for the holomorphic function of local chart ϕ composite with f, that is $\phi \circ f : \mathbb{D} \longrightarrow \mathbb{C}$. We shall denoted by λ_z the function $\lambda|\tilde{f}'(z)|$, and it is clear that the curvature inequality above will also hold for λ_z when ever the given metric has a curvature ≤ 4, except for the possible singularities.

Here come the Ahlfors' theorem:

Theorem 1 (Ahlfors [1])**.** *If the map $f : (\mathbb{D}, \mathrm{d}\sigma^2) \longrightarrow (M, \mathrm{d}s^2)$ is holomorphic, and M is a Riemann surface with negative curvature ≤ -4, then the inequality*

$$\mathrm{d}s \leq \mathrm{d}\sigma$$

will hold through out the disk.

Proof. Choose an arbitrary $R < 1$, set $v = \log R(R^2 - |z|^2)^{-1}$ for $|z| < R$. Denoted by $u = \log \lambda_z$, it suffices to show that $u \leq \lim_{R\to 1} v$. We note that $\triangle v = 4e^{2v}$ and consequently

$$\triangle(u - v) \geq 4\left(e^{2u} - e^{2v}\right)$$

Let us denote by E the open point set in $|z| < R$ for which $u > v$,It is clear that E cannot contain any singularities of λ_z, thus the inequality above is still valid in E, and $u - v$ is subharmonic[3] in E, thus has no maximum in E and must approach to the least upper bound on a sequence tending to the boundary of E. But E doesn't contain its boundary on $|z| = R$, for $v = +\infty$ whenever z tends to the $R-$ circle rather in the interior boundary points we have $u - v = 0$, thus $u \leq v$ is valid for all $|z| < R$, by letting $R \to 1$, here comes the theorem. ■

If the Riemann surface M itself is equipped with the hyperbolic metric, it is easily to have constant negative curvature -4, so result is to say the holomorphic map between Poincaré disk to any hyperbolic Riemann surface is Lipschitz with Lipschtiz constant 1, if we take the Riemann surface to be another Poincaré disk, then the result is reduced to the ordinary Schwarz's lemma.

Applications

As an application, the theorem can be used to investigate the numerical bound of Schottky's theorem, which states that if $f(z)$ is a holomorphic function from the unit disk to the complex plane $\mathbb{C}$, then the value $\log|f(z)|$ can be controlled on each small circle, that is there exists a $g(r)$, such that $\log|f(z)| < g(r)$ for all $|z| \leq r < 1$.

Here I shall just state the basic idea on how to handle this problem, the detail may refer to [1].

We equip the disk the hyperbolic metric to become a Poincaré disk, and we wish to give a hyperbolic metric on the plane, that entails to give a Riemann surface structure on the plane, and then the Riemannian structure, we consider

[2] It is quite convenient to consider the compact Riemann surface, for the Grassmanian $Gr_2(M)$ of a compact manifold is also compact, and the sectional curvature is a smooth map $K : Gr_2(M) \longrightarrow \mathbb{R}$,due to the mean-value theorem, if it has negative curvature, it must have a strictly negative upper bound.

[3] The definition of subharmonic function is very complicated, but I shall list it here:

A function $f : X \to \mathbb{R} \cup \{-\infty\}$ is called subharmonic if it upper-semicontinuous, i.e $\limsup_{x\to x_0} f(x) \leq f(x_0)$ for all $x_0 \in X$, and for any closed ball $\overline{B(x,r)}$ contained in X, there exists a real-valued function h on $\overline{B(x,r)}$ harmonic in $B(x,r)$ satisfies $f(y) \leq h(y)$ for all $y \in \partial B(x,r)$ and we have $f(y) \leq h(y)$ for all $y \in B(x,r)$

Here the criterion we used is the fact that if f is holomorphic, then $\log|f|$ is subharmonic.Of course, we still get the maximum module principle for subharmonic functions, that is the subharmonic function cannot achieve its maximum in the interior of its domain unless it is constant.

ζ_1 which maps the region outside of the segment $(0,1)$ onto the exterior of the unit disk such that $\zeta_1(\infty) = \infty$, $\zeta_1(1) = 1$, $\zeta_1(0) = -1$, explicitly, $\zeta_1(w)$ is given by

$$\zeta_1 + \zeta_1^{-1} = 4w - 2$$

then we set ζ_2, ζ_3, such that

$$\zeta_2(w) = \zeta_1\left(\frac{1}{w}\right) \quad \zeta_3(w) = \zeta_2(1-w)$$

Then these functions defines similar maps of the regions outside of the segments $(1,\infty), (-\infty, 0)$, now we introduce the complex structure on the plane

$$\Omega_1 = \{w \in \mathbb{C} : |w| \geq 1, |w-1| \geq 1\}$$

$$\Omega_2 = \{w \in \mathbb{C} : |w| \leq 1, |w| \leq |w-1|\}$$

$$\Omega_3 = \{w \in \mathbb{C} : |w| \leq 1, |w| \geq |w-1|\}$$

Define the Riemannian metric:

$$\mathrm{d}s_i = \frac{|\mathrm{d}\log\zeta_i|}{2(4+\log|\zeta_i|)} := \lambda_i |\mathrm{d}w|$$

By computation we know that it is the hyperbolic metric of a half-plane with negative constant -4, then by applying theorem 1, we have

$$\lambda|\mathrm{d}w| \leq \frac{|\mathrm{d}z|}{1-|z|^2}$$

By the geodesic calculation, we can find that the distance (in the sense of the hyperbolic metric) between $f(0)$ and $f(z)$ on the circle $|z| = r$ is less than

$$\frac{1}{2}\log\frac{1+r}{1-r}$$

Then by some analytic tricks, one will get the final result.

The Curvature of Compact Riemann Surfaces

It is well-known that the compact Riemann surfaces is classified by the genus g, for the case $g = 0$, it is the 1-projective plane $\mathbb{P}^1$ which is conformal with a sphere S^2, equipped with standard sphere metric

$$g = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$$

it gets the constant positive curvature 1. In this case of positive curvature, we call it the **elliptic** Riemann surface, there is only one case, that is the Riemann sphere.

For the case $g = 1$, it is the torus, we can give it the flat metric, i.e the inherited metric from the Euclidean space $\mathbb{C}$, it will get the constant curvature 0, the flat torus. In this case of flat curvature, we call it the **parabolic** Riemann surface.

For the case $g \geq 2$, we can equip it with Riemannian metric[4] so that it has constant (sectional) curvature -1, in this case we call it the **hyperbolic** Riemann surface.

So finally, let me give the Gauss-Bonnet formula on the compact Riemann surface to end this homework.

Let L be the holomorphic line bundle on the compact Riemann surface M, $\mathrm{d}s^2 = g\mathrm{d}z \otimes \mathrm{d}\bar{z}$ is the Hermitian metric on L, define a 2-form

$$\Theta = \frac{1}{4}\triangle \log g \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

Then we have the beautiful Gauss-Bonnet formula:

Theorem 2 (Gauss-Bonnet)**.** *Suppose D is the divisor on the compact Riemann surface M, g is the Hermitian metric on the holomorphic line bundle $L = \lambda(D)$, then we have*

$$\frac{i}{2\pi}\int_M \Theta = \deg(D) = \chi(L) - \frac{1}{2}\chi(M)$$

[4]The metric is stilled induced from the quotient, but differs from the case $g = 1$, the covering space of the Riemann surface with $g \geq 2$ is the Poincaré disk which is isometric with the $\mathbb{C}$ equipped with hyperbolic metric.

Reference

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