Symplectic Topology

An occasional note

Lecture 1. Equivariant Tubular Neighborhood Theorem.

1. Some Basic notions on Lie Groups Actions.

Let G be a Lie group, M be a smooth manifold, a Lie group action is a map:

$$G \longrightarrow \mathrm{Diff}(M),$$

or written in

$$G \times M \longrightarrow M$$
$$(g, x) \longmapsto g \cdot x$$

This is a left action, that is:

$$g \cdot (h \cdot x) = gh \cdot x$$

For a fixed $x \in M$, all the results acted by G is called the orbit of x, denoted by $\mathrm{orb}(x)$ or $G \cdot x$, i.e.:

$$\mathrm{orb}(x) = \{ gx \mid g \in G \}.$$

the collection of all the $g \in G$ who take x as an fixed point is called the Stablizer of x, denoted by $\mathrm{Stab}\, x$, i.e.

$$\mathrm{Stab}\, x = \{ g \in G \mid gx = x \}.$$

Although a Stablizer is a Lie subgroup of G, but usually not normal, again, the $\mathrm{orb}(x)$ usually not an embeded submanifold of M, but under the Compact Lie group action, it is an embeded submanifold.

Remark: the stablizers in a same orbit are conjugate to each other, i.e.

$$\mathrm{Stab}\, gx = g^{-1}\, \mathrm{Stab}\, x\, g$$

the proof is simple, just to show $g^{-1}\mathrm{Stab}\,x\,g \subset \mathrm{Stab}\,gx$ and $g\,\mathrm{Stab}\,gx\,g^{-1} \subset \mathrm{Stab}\,x$.

Suppose G acts on two manifolds M, N, a smooth map $\varphi: M \longrightarrow N$ is called an equivariant map if:

$$\forall x \in V,\ g \in G,\ \varphi(g.x) = g.\varphi(x),$$

which is to say, the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "g"'] & N \arrow[d, "g"] \\ M \arrow[r, "\varphi"'] & N \end{tikzcd}$$

Moreover, φ sends the orbit of x to the orbit of $\varphi(x)$.

Remark: If φ is an equivariant map, $\mathrm{Stab}\,x \subset \mathrm{Stab}\,\varphi(x)$.

For a fixed point $x \in M$, define an orbit mapping:

$$f_x : G \longrightarrow M$$
$$g \longmapsto g.x$$

the image of f_x is the $\mathrm{Orb}(x)$, consider the differential map at $1 \in G$:

$$(df_x)_1 : \mathfrak{g} \longrightarrow T_xM.$$

For each vector in Lie algebra $X \in \mathfrak{g}$, it induces a vector field $\tilde{X}$ on M:

$$\tilde{X}(x) := (df_x)_{\mathbb{1}} X.$$

this is called the fundamental vector field associated with X.

Remark: By our definition, the flow induced by $\tilde{x}(x)$ is: $\exp tX \cdot x$. The proof is simple:

Recall that a flow induced by $\tilde{X}(x)$ is a curve $\gamma(t)$ s.t.:

$$\tilde{X} \circ \gamma = \gamma'.$$

check that:

$$\begin{aligned}
\tilde{X} \circ \gamma(t) &= \left(df_{\exp tX \cdot x}\right)_{\mathbb{1}} X \\
&= \left.\frac{d f_{\exp tX \cdot x} \circ \exp sX}{ds}\right|_{s=0} \\
&= \left.\frac{d \exp sX \exp tX \cdot x}{ds}\right|_{s=0} \\
&= \left.\frac{d \exp(s+t) X \cdot x}{ds}\right|_{s=0} \\
&= \left.\frac{d \exp mX \cdot x}{dm}\right|_{m=t} = \gamma'(t).
\end{aligned}$$

Next, we consider the homogenous space $G/\mathrm{Stab}\,x$, f_x induced a natural map

$$\begin{aligned}
F_x : G/\mathrm{Stab}\,x &\longrightarrow M \\
[g] &\longmapsto gx
\end{aligned}$$

it is well-defined, since $[g'] = [g]$ if and only if $g'x = gx$.

Thm 1: $F_x : G/\mathrm{Stab}\,x \longrightarrow M$ is an injective immersion,

Proof: consider the tangent map of f_x at $\mathbb{1}\in G$:

$$(df_x)_{\mathbb{1}} : \mathfrak{g} \longrightarrow T_xM.$$

the kernel of $(df_x)_{\mathbb{1}}$ is the set:

$$\{X \in \mathfrak{g} : \tilde{X} \equiv 0\}.$$

which implies, if $X \in \ker(df_x)_{\mathbb{1}}$, we have:

$$\frac{d \exp tX \cdot x}{dt} = 0 \quad \text{for all } t,$$

$\Rightarrow$ $\exp tX \cdot x$ is a constant curve, and

$$\exp tX \cdot x = \exp 0X \cdot x = x$$

$\Rightarrow$ X is in $L(\mathrm{Stab}\,x)$.

By the commutative diagram:

$$\begin{tikzcd}
\mathfrak{g} \arrow[d, "(d\pi)_{\mathbb{1}}"'] \arrow[dr, "(df_x)_{\mathbb{1}}"] & \\
T_{[\mathbb{1}]}(G/\mathrm{Stab}\,x) \arrow[r, "(d\bar{F}_x)_{\mathbb{1}}"'] & T_xM
\end{tikzcd}$$

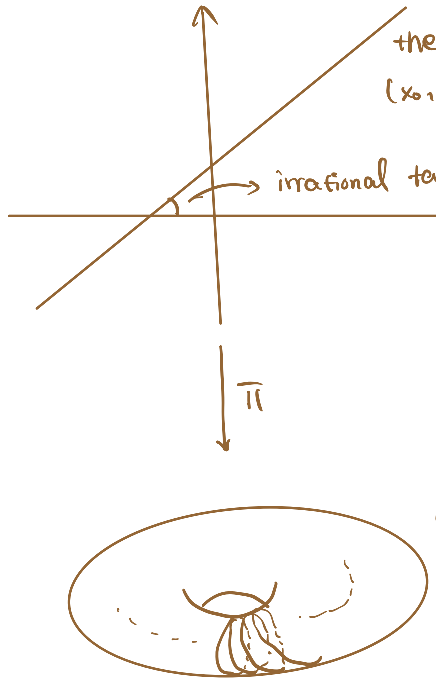
we have: $\ker(d\bar{F}_x)_{\mathbb{1}} = (d\pi)_{\mathbb{1}} \ker(df_x)_{\mathbb{1}} = (d\pi)_{\mathbb{1}} L(\mathrm{Stab}\,x) = 0$

thus it is an immersion.

Remark: Although the image of f_x is the orbit of x $\mathrm{orb}(x)$, but $\mathrm{orb}(x)$ may be not a submanifold of M, for example: let $GL_1(\mathbb{R}) = \mathbb{R}^*$ acts on $S^1\times S^1$

$$t\cdot(x,y) \longmapsto (x+t,\ x+\alpha t)$$

for some fixed $\alpha \in \mathbb{Q}^c$.



For the topology exhibited from $F_x(\mathbb{R}^*/\mathrm{Stab}_x)$ is not same as the original one in T^2,

Thm 2: Any orbit of a compact Lie group action is a submanifold.

Proof: Recall that a proper injective imersion is an embeding, thust just show that F_x is in this case a proper one.

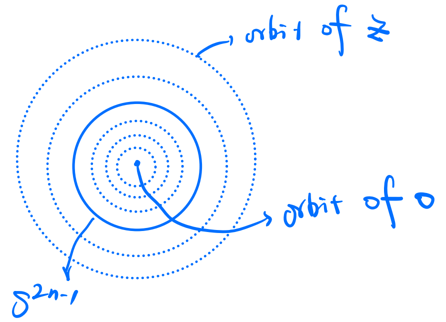
$$\begin{tikzcd} G \arrow[d, "\pi"'] \arrow[dr, "f_x"] & \\ G/\mathrm{Stab}\, x \arrow[r, "F_x"'] & M \end{tikzcd}$$

For a compact $K \subset M$, $F_x^{-1}(K) = \pi \circ f_x^{-1}(K)$, thus compact

Since f_x is continuous, $f_x^{-1}(K)$ is closed in G, since π is a closed map, $\pi(f_x^{-1}(K))$ is compact

Examples: 1. S^1 acts on $\mathbb{C}^n$ by complex multiplication:

$$e^{i\theta} \cdot (z_1, \cdots z_n) := (e^{i\theta} z_1, \cdots, e^{i\theta} z_n).$$



Except 0 as the fixed point, all the other orbits are circles, the sphere S^{2n-1} is stable under this action,

An G-action is said to be effective, if each element moves at least one x in M, i.e.: $\forall g \in G, g \neq \mathbb{1}$, $\exists x \in M$ s.t. $gx \neq x$, also equivalent to:

$$\bigcap_{x \in M} \mathrm{Stab}\, x = \{1\}.$$

(For if $g \in \bigcap_{x \in M} \mathrm{Stab}\, x \Leftrightarrow gx = x, \forall x \in M \Leftrightarrow$ G-action is not effective, unless $g = \mathbb{1}$.)

Proposition: The group $\bigcap_{x \in M} \mathrm{Stab}\, x$ is a closed normal subgroup in G, G action on M will induce an effective action of $G / \bigcap_{x \in M} \mathrm{Stab}\, x$ on M.

Proof: $\bigcap_{x \in M} \mathrm{Stab}\, x$ is clearly a Lie subgroup of G, for $g \in \bigcap_{x \in M} \mathrm{Stab}\, x$, $\forall h \in G$, for any x,

$$h^{-1} g h x = h^{-1} g (hx) = h^{-1} h x = x$$

$$\Rightarrow h^{-1} g h \in \bigcap_{x \in M} \mathrm{Stab}\, x \text{, thus a normal subgroup.}$$

Let: $[g].x := gx$, it clearly well-defined, and effective.

Example: 1. S^1 acts on $S^3 \subset \mathbb{C}^2$ via:

$$t(z_1, z_2) := (t^{m_1} z_1, t^{m_2} z_2), \quad m_1, m_2 \in \mathbb{Z}.$$

it is effective, iff $(m_1, m_2) = 1$

Proof: If for some $t \in S^1$ s.t:

$$(t^{m_1} z_1, t^{m_2} z_2) = (z_1, z_2)$$

for all $z_1, z_2 \in S^3$, which if and only if:

$$t^{m_1} = t^{m_2} = 1 = e^{2k\pi i}$$

thus $t = e^{\frac{2k\pi i}{m_1}} = e^{\frac{2n\pi i}{m_2}}$ where $k = 1, \cdots, m_1 - 1$
$n = 1, \cdots, m_2 - 1$

$\Rightarrow k = n \frac{m_1}{m_2}$.

WLoG, let $m_1 < m_2$, since $m_2 \nmid n \Rightarrow m_1$ must containing a factor of m_2, i.e. $(m_1, m_2) \neq 1$.

2. If G is simple, all the action is effective, or trivial, such as SO_3.

The last thing is about the orbit space.

Define an equivalent relation on M:

$$x \sim_G y \iff x, y \text{ are in a same } G\text{-orbit},$$

the quotient space is denoted by M/G, the orbit space.

the quotient space may be even not a Hausdorff space,

Example: $\mathbb{C}^* = GL_1(\mathbb{C})$ act by multiplication on $\mathbb{C}^{n+1}$,

$$\lambda . z = \lambda z.$$

then the only nbghd containing $[0]$ is $\mathbb{C}^{n+1}/\mathbb{C}^*$, thus not a

Hausdorff one:

Proof: The orbit of $\mathbb{C}^*$ is the set:

$$\{\lambda z : \lambda \in \mathbb{C}^*\},$$

thus a nbghd of $[0]$ is the collection:

$$\mathcal{U} = \Big\{ \{\lambda z : \lambda \in \mathbb{C}^*\} : z \in U_0 \Big\}$$

where U_0 is the nbghd of 0 in $\mathbb{C}^{n+1}$, consider the $\pi^{-1}(\mathcal{U})$

$$\pi^{-1}(\mathcal{U}) = \bigcup_{z \in U_0} \{\lambda z : \lambda \in \mathbb{C}^*\} \subset \mathbb{C}^{n+1},$$

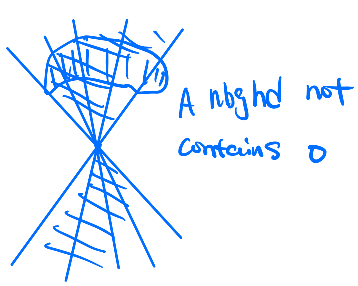
for any $z_0 \in \mathbb{C}^{n+1} \setminus \{0\}$, there exists an $z \neq 0$ in U_0, such that:

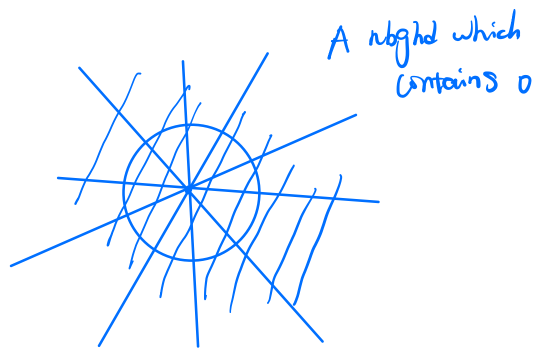
$$\lambda = \frac{z_0}{z} \in \mathbb{C}^*$$

$$\Rightarrow z_0 \in \pi^{-1}(\mathcal{U}) \Rightarrow \pi^{-1}(\mathcal{U}) = \mathbb{C}^{n+1} \Rightarrow \mathcal{U} = \mathbb{C}^{n+1}/\mathbb{C}^*,$$

thus not a Hausdorff Space.

However $\left(\mathbb{C}^{n+1}/\mathbb{C}^*\right) \setminus \{[0]\}$ is the Hausdorff one, which is $\mathbb{C}P^n$,





Thm: If G is a compact Lie group, M/G is Hausdorff, the Quotient map $\pi: M \longrightarrow M/G$ is a proper and closed mapping.

Proof: First, π is an open map, for if U is open in M, to check if $\pi(U)$ an open set, just check that:

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g \cdot U$$

it is a union of open sets, thus $\pi^{-1}(U)$ is open.

If G is compact, π is also closed, for any closed subset $A \subseteq M$, $\pi(A)$ is closed iff $\pi^{-1}(\pi(A))$ is closed,

$$\pi^{-1}(\pi(A)) = \theta(G \times A)$$

where θ is the action map:

$$\theta: G \times M \longrightarrow M.$$

thus it suffices to show θ is a closed map, whenever G is compact.

Let $C \subseteq G \times M$ be any closed subset, $y \in \overline{\theta(C)}$, then $\exists$ sequence $\{g_n\}, \{x_n\}$ such that:

$$y = \lim_{n \to} g_n \cdot x_n.$$

Since G is compact, thus $g_n \longrightarrow g \in G$, thus:

$$x_n = g_n^{-1} \cdot g_n x_n \longrightarrow g^{-1}(y)$$

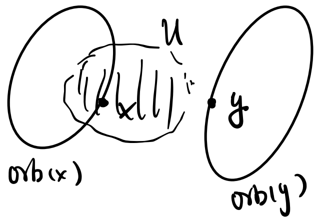
thus: $(g_n, x_n) \longrightarrow (g, g^{-1}(y)) \in C$, thus:

$$y = g \cdot g^{-1} y \in \theta(C) \Rightarrow \pi \text{ is closed.}$$

Since π is closed, fibers of π are compact, hence π is proper.

Finally, to show M/G is a Hausdorff space;

Let $orb(x)$, $orb(y)$ be two distinct orbits, they're compact, since M is Hausdorff, $\exists\, U$, a neiborghood of x, s.t: $\overline{U} \cap orb(y) = \emptyset$. then



$\pi(U)$, $\pi(\overline{U})$ are both open, they are not intersect, contains $\pi(x)$, $\pi(y)$.

2. Equivariant Tubular Neighborhood Theorem

Now, we consider the Compact Lie Group action $G \times M \longrightarrow M$, in this case the orbit $orb(x)$ is an embeded submanifold of M.

The usual tubular nbghd Thm state the relation between the original manifold and the normal bundle;

If $N \overset{i}{\hookrightarrow} M$ is the embeded submanifold of M, for each $x \in N$, define the normal space:

$$N_x := T_x M / T_x N$$

Then:

$$\mu_N := \coprod_{x \in N} N_x$$

is the normal bundle of N, then N will be the embeded submanifold of μ_N via the zero section:

$$N \overset{i_0}{\hookrightarrow} \mu_N$$

$$x \longmapsto (x, 0)$$

then there exists a $\underbrace{\text{nbghd}}_{U}$ of N in μ_N and a $\underbrace{\text{nbghd}}_{V}$ of N in M and a diffeomorphism $\varphi: U \longrightarrow V$ s.t the diagram commutes:

$$\begin{tikzcd}
N \arrow[r, hook, "i_0"] \arrow[dr, "i"'] & U \subset \mu_N \arrow[d, "\varphi"] \\
& V \subset M
\end{tikzcd}$$

Here, we take $N = orb(x)$, Let $V_x = T_x M / T_x orb(x)$ be the normal space.

The normal bundle is endowed with a G-action:

$$G \times orb(x) \times V_x \longrightarrow orb(x) \times V_x.$$

$$(g, (y, [X])) \longmapsto (gy, [(dg)_x X])$$

Now, we wonder whether exists an equivariant diffeomorphism φ from some equivariant tubular nbghd to another.

Thm: There exists an equivariant nbghd $\mathcal{U}$ of $orb(x)$ in $orb(x) \times V_x$, a nbghd U of $orb(x)$ in M, and an equivariant diffeomorphism:

$\varphi: U \longrightarrow V$ s.t.

$$\begin{tikzcd}
orb(x) \arrow[r, hook, "i_0"] \arrow[dr, "i"'] & U \subset orb(x) \times V_x \arrow[d, dotted, "\varphi"] \\
& V \subset M
\end{tikzcd}$$

commutes.

Proof: **Lemma 1:** If M equipped with a G-invariant metric, then, the exponential map $exp: W \subset T_x M \longrightarrow exp(W) \subset M$ is an equivariant map.

Proof: All what we need to prove is the following diagram commutes:

$$\begin{tikzcd} T_xM \arrow[r, "(dg)_x"] \arrow[d, "\exp"'] & T_{gx}M \arrow[d, "\exp"] \\ M \arrow[r, "g"'] & M \end{tikzcd}$$

Or just:

$$g\circ \exp tX = \exp (dg)_x tX$$

First notice that, since $\exp tX$ is a geodesic, g is an isometry thus LHS is a geodesic,

$$RHS = \exp t(dg)_x X$$

which is also a geodesic, pass the initial point gx, with initial velocity $(dg_x)X$, now, let $\gamma(t) = g\circ \exp tX$,

$$\gamma(0) = g\circ \exp 0 = gx,$$

$$\gamma'(0) = \frac{d\, g\circ \exp tX}{dt}\Big|_{t=0} = (dg)_x\left(\frac{d\exp tX}{dt}\Big|_{t=0}\right)$$
$$= dg_x X$$

According to the uniqueness of geodesic, the lemma is proved.

Now, we wish to equip a G-invariant metric on M, then, we can identify the normal space V_x with the orthogonal complement: $T_x\mathrm{Orb}(x)^{\perp}$,

Since the exponential map is a diffeomorphism from a nbghd of M in TM to M:

$$\exp: W\subset TM \longrightarrow M$$

We can choose $\mathcal{U} = W \cap \mu_{orb(x)}$, which is a nbgh'd of M in $\mu_{orb(x)}$ and:

$$\exp\big|_{\mathcal{U}} : \mathcal{U} \subset \mu_{orb(x)} \longrightarrow \exp(\mathcal{U})$$

is an equivariant diffeomorphism by the previous lemma, so we just need to prove the existence of the G-invariant metric.

Lemma: If G is a compact Lie group acts on M, then there exists a G-invariant metric on M.

Proof: Choose $\omega \in \Omega^n(G)$, where $n = \dim G$, let:

$$\omega_G = g \cdot \omega \quad , g \in G.$$

It is a left invariant n-form on G, define:

$$|G| = \int_G \omega_G .$$

for any metric $\langle \cdot, \cdot \rangle$ defined in M, define:

$$\{X, Y\} := \frac{1}{|G|} \int_G \langle dg_x X, dg_x Y \rangle \, \omega_G .$$

it is G-invariant, for any $h \in G$:

$$\{dh_x X, dh_x Y\} = \frac{1}{|G|} \int_G \langle dg_{\cancel{hx}} dh_x X, dg_{hx} dh_x Y \rangle \, \omega_G$$

$$= \frac{1}{|G|} \int_G \langle dg_{hx} dh_x X, dg_{hx} dh_x Y \rangle \, L_{h^{-1}}^* \omega_G$$

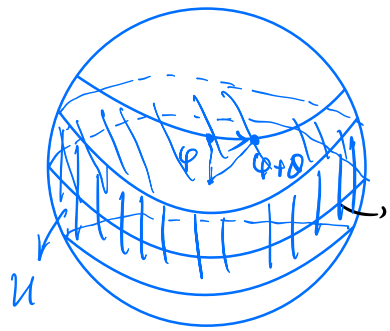
$$= \frac{1}{|G|} \int_G \langle dg_x X, dg_x Y \rangle \, \omega_G$$

$$= \{X, Y\} .$$

That ends the proof.

Example: S^1 acts on S^2 by rotation:

$$\theta : (\varphi, h) \longmapsto (\varphi+\theta, h)$$



Then every orbit is the weft, choose the equator circle as the orbit of $(\varphi, 0) := x$, then the tubular nbghd of Gx is ~~the~~ whole normal bundle: $S^1 \times \mathbb{R}$ — a cylinder, a tubular of Gx in S^2 is the set:

$$U = \left\{ (\varphi, h) \mid \varphi \in [0, 2\pi], |h| < \frac{1}{2} \right\}.$$

the equivariant diffeomorphism:

$$\varphi : S^1 \times \mathbb{R} \longrightarrow U$$
$$(\theta, x) \longmapsto \left(\theta, \frac{x}{2\sqrt{1+x^2}}\right)$$