On the Symplectic Structures on the Cotangent Bundles

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Abstract

The cotangent bundle $M=T^*X$ of a smooth manifold X has a natural symplectic structure $\omega=\Sigma_{i=1}^n dx_i\wedge d\xi_i$, in this paper, we aim to investigate two things of the cotangent bundle, the first is about the symplectomorphisms between them, as the main result we proved that a symplectomorphisms $g:T^*X_1\longrightarrow T^*X_2$ is a lifting of a diffeomorphism $f:X_1\longrightarrow X_2$ if and only if it preserves the Liouville 1-form, the second thing is about the Lagrangian submanifold of the cotangent bundle, we gave some examples, the images of the sections and the conormal bundles, as an application, we showed that a diffeomorphism between two manifolds $\phi:M_1\longrightarrow M_2$ is a symplectomorphisms if and only if the graph Γ_ϕ is a Lagrangian submanifold of the product manifold $M_1\times M_2$.

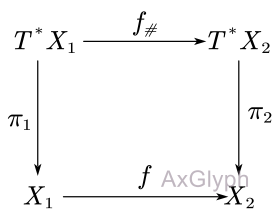
Keywords

Symplectic Manifolds, Cotangent Bundles, Pull-back, Conormal Bundles, Liouville 1- form.

1.Introduction

Let X be an n-dimensional manifold, $M=T^*X$ is the cotangent bundle, choose a coordinate chart $(U,x_1,\ldots,x_n)$ of $x\in U$, then $dx_1,\ldots,dx_n$ is a basis of the cotangent space T_x^*X, if $\xi\in T_x^*X$, then $\xi=\Sigma_{i=1}^n\xi_i dx_i$ for some real coefficients $\xi_i\in\mathbb{R}$, that induces a map $T^*U\rightarrow\mathbb{R}^{2n}$ via $(x,\xi)\mapsto(x_1,\ldots,x_n,\xi_1,\ldots,\xi_n)$, thus the cotangent bundle M is a $2n-$dimensional manifold, we define a 2-form ω associated with the local coordinate $(T^*U,x_1,\ldots,x_n,\xi_1,..,\xi_n)$ by $\omega=\Sigma_{i=1}^n dx_i\wedge d\xi_i$, it is a canonical symplectic form[1] on M which makes the cotangent bundle be a symplectic manifold [1], define the 1-form $\alpha=\Sigma_{i=1}^n\xi_i dx_i$ which is the Liouville 1-form, clearly $\omega=-d\alpha$, let $\pi:T^*X\longrightarrow X$ be the natural projection via π: $p=(x,\xi)\mapsto x$, then one can define the Liouville 1-form pointwisely by $\alpha_p=(d\pi)_p^*\xi=\xi\circ(d\pi)_p\in T_p^*M$ [1][2], where $(d\pi)_p^*$ stands for the transpose of the tangent map.

Let X_1,X_2 be 2 diffeomorphic $n-$dimensional manifolds with cotangent bundle M_1,M_2 respectively, α_1,α_2 are the Liouville 1-forms on their cotangent bundles respectively, f: $X_1\longrightarrow X_2$ is a diffeomorphism, then there is a natural diffeomorphism $f_\#$: $M_1\longrightarrow M_2$ which lifts f, namely, if $(x_1,\xi_1)\in M_1$, then $f_\#(x_1,\xi_1)=(x_2,\xi_2)$, where $f(x_1)=x_2$, $\xi_1=(df)_{x_1}^*\xi_2$, and the following diagram commutes



[1] If without of an extra explanation, we shall use the notion ω to represent for the canonical symplectic form on a manifold, and α the Liouville 1-form on the cotangent bundle.

Moreover, if $g:X_2 \longrightarrow X_3$ is another diffeomrphism, then one has $(g\circ f)_\# = g_\# \circ f_\#$, which implies that the T^* is a covariant functor from the category of manifolds to the category of symplectic manifolds.

About the lifting $f_\#$ we know the following result from [1-3]:

Lemma 1.1 The lifting $f_\#$ is a symplectomorphism which presrves the Liouville 1-form, i.e. $f_\#^*\alpha_2 = \alpha_1$, where $f_\#^*:\Omega^1(T^*X_2) \longrightarrow \Omega^1(T^*X_1)$ is the pull-back [4] of $f_\#$.

So now, there is a natural question, do all symplectomorphisms come from the lifting of some diffeomrhisms ? And since the cotangent bundle is now a symplectic manifold, what does its Lagrangian submanifolds look like ? Now we shall answer these questions.

2.The Symplectomorphisms Between Cotangent Bundles

The answer of the first question is negative, for the lemma 1.1 tells us that if the symplectomorphisms between the cotangent bundles is a lifting of some diffeomorphisms, it should at least preserve the Liouville 1-form, however, there do exists some symplectomorphisms which fail to hold for this property.

Example Let $g\in C^\infty(X)$, and $p:T^*X \longrightarrow T^*X$ by $p(x,\xi) = (x,\xi + dg_x)$, here dg_x represents for the total differential of g at $x\in X$, then by calculation one can find that $p^*\alpha = \alpha + \pi^* dg$ and p is a symplectomorphisms.

So, what is the criterion for symplectomorphisms arise as lifts of diffeomorphisms? We have the following result:

Theorem 2.1 Let $g:T^*X_1 \longrightarrow T^*X_2$ be a symplectomorphism between two cotangent bundles, g arises as a lift of diffeomorphism $f:X_1 \longrightarrow X_2$ if and only if g preserves the Liouville 1-forms, i.e. $g^*\alpha_2 = \alpha_1$.

The "only if" part is the statement of lemma 1.1, so it suffices to prove the "if" part. To be convenient, we shall just study the case $X_1 = X_2 = X$, If we denoted by $g(x_1,\xi_1) = (x_2,\xi_2)$, it is natural to write $f(x_1) = x_2$ to gain the commutative diagram, but we do not know whether this is a diffeomorphism yet, here we need some lemmas.

Lemma 2.2 There exists a unique vector field v on T^*X such that the interior product $\iota_v\omega = -\alpha$.

Proof: By comparing the coefficients, one can find the unique vector field is $v = \Sigma_{i=1}^n \xi_i \partial_{\xi_i}$, which is expressed in the local chart $(T^*U, x_1,\ldots,x_n,\xi_1,..,\xi_n)$. ■

Now we consider the phase curve $\gamma(t) = \exp tv(p):\mathbf{R} \longrightarrow T^*X$ induced by the vector field v, that is the parametrized curve with the initial condition $\gamma(0) = p$ and satisfies the differential equation $v\circ\gamma = \gamma'$, we denoted by $\exp tv$: $T^*X \longrightarrow T^*X$ the phase flow induced by v, which is a one-parameter group on T^*X by sending p to $\exp tv(p)$.

Lemma 2.3 If g is a symplectomorphism preserves the Liouville 1-form α, then g commutes with the phase flow induced by v, i.e. $g\circ \exp tv = \exp tv \circ g$.

Proof: Recall that the push-forward map $g_*:T_p(T^*X) \longrightarrow T_{g(p)}(T^*X)$ of g acts on v is defined as $(g_* v_p)_{g(p)} = (dg)_p v_p$, then the phase curve induced by $g_* v$ at point $g(p)$ is

$$(g\circ\gamma)(t) = g\circ \exp tv(g(p))$$

Thus the phase curve passed through the point p should be $g\circ \exp tv(g^{-1}(p))$, thus the phase flow induced by $g_* v$ is exactly $g \circ \exp tv \circ g^{-1}$, then notice that

$$\begin{aligned} g^*\alpha(v_p) &= \alpha((dg)_p v_p) = g^*(-\iota_v \omega(v_p)) \\ &= -\iota_v \omega((dg)_p v_p) = g^*(\omega_p(v_p, v_p)) \\ &= \omega((dg)_p v_p, v_p) = 0 \end{aligned}$$

Which implies that $g_* v = v$, thus $g \circ \exp tv = \exp tv \circ g$. ■

Lemma 2.4 $\exp tv(x, \xi) = (x, e^t \xi)$.

Proof: Choose a local coordinate $(T^* U, x_1, ..., x_n, \xi_1, .., \xi_n)$, and assume that phase curve can be expressed as

$$\gamma(t) = \exp tv(p) = (f_1(t), ..., f_n(t), g_1(t), ..., g_n(t))$$

Then the restriction of v on γ is $(v \circ \gamma)(t) = \Sigma_{i=1}^n g_i(t) \partial_{\xi_i}$, by definition we have the differential equations $f_i'(t) = 0, g_i'(t) = g_i(t)$, and the initial condition $f_i(0) = x_i, g_i(0) = \xi_i$, thus we can get the solution $f_i(t) = x_i,\ g_i(t) = e^t \xi_i$. ■

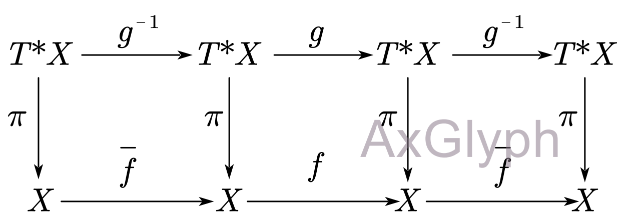
Lemma 2.5 If we denoted by $g(x_1, \xi_1) = (x_2, \xi_2)$, then $g(x_1, \lambda\xi_1) = (x_2, \lambda\xi_2)$ for all $\lambda \in \mathbb{R}$.

Proof: Applying lemma 2.3 and 2.4, one has

$$\begin{aligned} g(x_1, \lambda\xi_1) &= (g \circ \exp \log \lambda v)(x_1, \xi_1) \\ &= ((\exp \log \lambda v) \circ g)(x_1, \xi_1) \\ &= (\exp \log \lambda v)(x_2, \xi_2) = (x_2, \lambda\xi_2) \qquad \blacksquare \end{aligned}$$

Now we can prove our main result.

The proof of theorem 2.1: We denoted by $g(x_1, \xi_1) = (g_1(x_1, \xi_1), g_2(x_1, \xi_1))$, where g_i are both smooth, applying lemma 2.5 we have $g_1(x_1, \lambda\xi_1) = g_1(x_1, \xi_1)$ and $g_2(x_1, \lambda\xi_1) = \lambda g_2(x_1, \xi_1)$, by letting $\lambda = 0$, we gain that $g_1(x_1, \xi_1) = g_1(x_1, 0)$ and $g_2(x_1, 0) = 0$, which means that g_1 only acts on the X part, g_2 only acts on the fiber parts, thus we have $f(x_1) = x_2 = g_1(x, 0)$, which is obviously smooth. The bijection comes from the commutative diagram:



And the smmothness of the inverse map is obviously.

Finally, we need to show that the g is exactly the lifting of f, it suffices to show that $\xi_1 = (df)_x^* \xi_2$, in convenient we denoted by $g(p) = q$ insetead of the former notion, to see this, we differentiate both side of the identity $f \circ \pi = \pi \circ g$ and gain $(d\pi)_q \circ (dg)_p = (df)_x \circ (d\pi)_p$, since g preserves the Liouville 1-form, we have $(dg)_p^* \alpha_q = \alpha_p$, and by the definition we have:

$$\begin{aligned} (dg)_p^* (d\pi)_q^* \xi_2 = (d\pi)_p^* \xi_1 &\Rightarrow ((d\pi)_q \circ (dg)_p)^* \xi_2 = (d\pi)_p^* \xi_1 \\ &\Rightarrow (d\pi)_p^* (df)_x^* \xi_2 = (d\pi)_p^* \xi_1 \end{aligned} \tag{1.1}$$

Consider a smooth section of the cotangent bundle $s:X \longrightarrow T^*X$, thus we have $\pi \circ s = \mathrm{id}_X$, again, differentiate of both side, we have $(ds)_x^*(d\pi)_p^* = \mathrm{id}_{T_xX}$, by applying $(ds)_x^*$ on both side of (1.1), we then got the desired conclusion. ■

We can conclude a corollary through the proof.

Corollary 2.6 If g is a symplectomorphism which preserves the Liouville 1-form, then the matrix of the tangent map $(dg)_p$ expressed in the local coordinate of $p = (x_1, \xi_1)$ is a blocked diagonalized matrix, i.e.

$$(dg)_p = \begin{pmatrix} A & \\ & B \end{pmatrix}$$

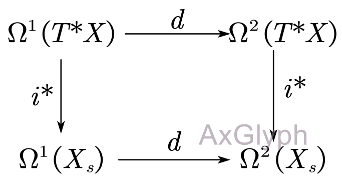
Thus the lifting of the diffeomorphism is unique.

3.Lagrangian Submanifolds of the Cotangent Bundles

In this part, we will study the Lagrangian submanifolds of the cotangent bundles, without of an extra explanation, all submanifolds are referring to the closed embedded submanifolds.

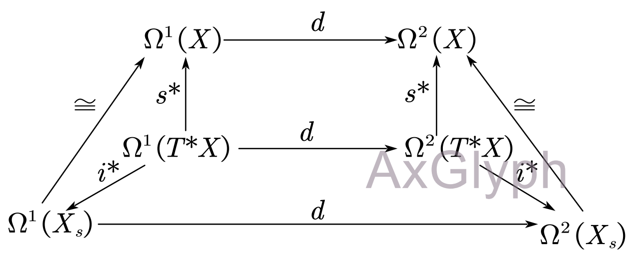
Recall that a submanifold Y of a $2n$-dimensional symplectic manifold (M, ω) is Lagrangian, if for each $p \in Y$, T_pY is a Lagrangian subspace of T_pM, i.e. $\omega_p|_{T_pY} = 0$ and $\dim T_pY = 1/2 \dim T_pM$, which is equivalent to say if $i: Y \hookrightarrow M$ is the inclusion map, $i^*\omega = 0$ and $\dim T_pY = n$, thus the Lagrangian is an n-dimensional submanifold.

We first introduce the image of some section $X_s = \mathrm{Im}\, s = \{(x, \xi_s) \in T^*X : (x, \xi_s) = s(x)\}$, where $s: X \longrightarrow T^*X$ is the section of the cotangent bundle, which is a 1-form in $\Omega^1(X)$, X_s is an n-dimensional submanifold of T^*X where the intersection with T^*U is given by $\xi_1 = \cdots = \xi_n = \xi_s$, let $i: X_s \hookrightarrow T^*X$ be the inclusion map, then $\pi \circ i: X_s \longrightarrow X$ is a diffeomorphism, moreover, the pull-back of i is a chain map [4], i.e. we have the following diagram commutes:



Theorem 3.1 X_s is Lagrangian if and only if s is a closed 1-form in $\Omega^1(X)$.

Proof: s will induce a pull-back, namely, $s^*: \Omega^*(T^*X) \longrightarrow \Omega^*(X)$, and particularly, since $s: X \longrightarrow X_s$ is a diffeomorphic, this induces the isomorphism between $\bar{s}^*: \Omega^*(X_s) \longrightarrow \Omega^*(X)$, and we have the following diagram commutes:



Next we claim that $s^*\alpha = s$, in fact, we denoted by $p = (x, \xi_s)$, then by definition:

$$(s^*\alpha)_x = (ds)_x^*(d\pi)_p^*\xi_s = (d(\pi \circ s))_x^*\xi_s$$
$$= \xi_s = s(x)$$

Hence $s^*\alpha = \alpha$, then by the diagram chasing, the X_s is Lagrangian, if and only if $di^*\alpha = 0$, if and only if $ds^*\alpha = 0$ (due to the isomorphism), if and only if $ds = 0$, that is a closed form. ■

Whenever X is simply connected, we know from topology that the 1st de Rham cohomology group is 0, thus all such Lagrangian submanifolds have the form X_{df}, such a primitive function f is called the generating function, two functions generate the same Lagrangian if they differ by a constant.

Corollary 3.2 If X is a compact manifold, then the cardinality $|X_{df} \cap X_0| \geq 2$, where X_0 stands for the zero section.

Proof: The elements in the set $X_{df} \cap X_0$ is exactly the critical points of f, and since X is compact, it is clearly that f has at least two critical points. ■

However, the Lagrangian submanifolds are not all the image of closed 1-form, as an example, we shall introduce the conormal bundle.

The conormal space at $y \in Y$ is the set $N_y^*Y = \{\xi \in T_y^*X : \xi(v) = 0, \forall v \in T_yY\}$, that is the collection of those cotangent vectors which have no T_y^*Y components, the conormal bundle of Y is

$$N^*Y = \coprod_{y \in Y} N_y^*Y$$

This is an n-dimensional submanifold of T^*X, because when we choose the adapted coordinate chart U to Y on X, $N^*(Y \cap U)$ is the coordinate chart on the conormal bundle, by some simply observations:

Proposition 3.3[1] The inclusion $i: N^*Y \hookrightarrow T^*X$ pulls the Liouville 1-form back to 0, i.e., $i^*\alpha = 0$, hence N^*Y is Lagrangian.

For example, if we take $Y = \{x\}$, then the conormal bundle is a line bundle, that is the cotangent fiber T_xX, if we take $Y = X$, then the conormal bundle is the zero section X_0.

4.Some Applications to Symplectomorphisms

The Lagrangian submanifolds can be used to study the symplectomorphisms, let $(M_1, \omega_1), (M_2, \omega_2)$ be two $2n$-dimensional diffeomorphically symplectic manifold, $\phi: M_1 \longrightarrow M_2$ is the diffeomorphism, there is a analytical way to check if it is a symplectomorphism, that is by computing the matrix of the differential map.

Proposition 4.1 The diffeomorphism ϕ is the symplectomorphism, if (and only if) the matrix of $(d\phi)_{p_1}$ under the canonical coordinate satisfies

$$(d\phi)_{p_1}^T \begin{pmatrix} & I \\ -I & \end{pmatrix} (d\phi)_{p_1} = \begin{pmatrix} & I \\ -I & \end{pmatrix}$$

The proof is by directly computing, next we shall introduce a geometric interpretation of checking the symplectomorphisms.

For $(p_1,p_2)\in M_1\times M_2$, define two natural projections $\pi_i:M_1\times M_2\longrightarrow M_i$, by sending (p_1,p_2) to p_i, where $i=1,2$, then $\omega=\pi_1^*\omega_1+\pi_2^*\omega_2$ is closed 2-form on the product manifold, because

$$\begin{aligned}d\omega&=d\pi_1^*\omega_1+d\pi_2^*\omega_2\\&=\pi_1^*d\omega_1+\pi_2^*d\omega_2=0\end{aligned}$$

And it is also symplectic, since

$$\omega^{2n}=\binom{2n}{n}(\pi_1^*\omega_1)^n\wedge(\pi_2^*\omega_2)^n\neq0$$

Hence symplectic, moreover, if $\lambda,\mu\neq0$, the linear combination $\lambda\pi_1^*\omega_1+\mu\pi_2^*\omega_2$ is also symplectic, so it allows us to define the twisted product form $\tilde{\omega}=\pi_1^*\omega_1-\pi_2^*\omega_2$.

Recall that the graph of ϕ is defined as $\Gamma_\phi=\{(p,\phi(p)):p\in M_1\}$, which is a submanifold of the product manifold, we note that $\gamma(p)=(p,\phi(p)):M_1\longrightarrow\Gamma_\phi$ is the diffeomorphism, thus Γ_ϕ is an n-dimensional submanifold.

Theorem 4.2 The diffeomorphism ϕ is a symplectomorphism, if and only if the graph Γ_ϕ is the Lagrangian of the twisted product manifold $(M_1\times M_2,\tilde{\omega})$.

Proof: The graph Γ_ϕ is Lagrangian, if and only if $\gamma^*\tilde{\omega}=0$, since

$$\begin{aligned}\gamma^*\tilde{\omega}&=\gamma^*\pi_1^*\omega_1-\gamma^*\pi_2^*\omega_2\\&=(\pi\circ\gamma)^*\omega_1-(\pi_2\circ\gamma)^*\omega_2\qquad\Longrightarrow\qquad\gamma^*\tilde{\omega}=0\Leftrightarrow\phi^*\omega_2=\omega_1\\&=\mathrm{id}_{M_1}^*\omega_1-\phi^*\omega_2\end{aligned}$$

■

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