

Symplectic Topology

An occasional note

Lecture 1. Equivariant Tubular Neighborhood Theorem.

1. Some Basic notions on Lie Groups Actions.

Let G be a Lie group, M be a smooth manifold, a Lie group action is a map:

$$G \longrightarrow \text{Diff}(M),$$

or written in

$$G \times M \longrightarrow M$$

$$(g, x) \longmapsto g \cdot x$$

This is a left action, that is:

$$g \cdot (h \cdot x) = gh \cdot x$$

For a fixed $x \in M$, all the results acted by G is called the **orbit** of x , denoted by $\text{orb}(x)$ or $G \cdot x$, i.e.:

$$\text{orb}(x) = \{g \cdot x \mid g \in G\}.$$

the collection of all the $g \in G$ who take x as an fixed point is called the **Stabilizer** of x , denoted by $\text{stab } x$, i.e.:

$$\text{stab } x = \{g \in G \mid g \cdot x = x\}.$$

Although a stabilizer is a ^{Lie} subgroup of G , but usually not normal, again, the $\text{orb}(x)$ usually not an ^{embedded} submanifold of M , but under the Compact Lie group action, it is an embedded submanifold.

Remark: the stabilizers in a same orbit are conjugate to each other, i.e.

$$\text{Stab } gx = g^{\dagger} \text{Stab } x g$$

the proof is simple, just to show $g^{\dagger} \text{Stab } x g \subset \text{Stab } gx$ and $g \text{Stab } x g^{\dagger} \subset \text{Stab } x$.

Suppose G acts on two manifolds M, N , a smooth map $\varphi: M \rightarrow N$ is called an equivariant map if:

$$\forall x \in V, g \in G, \varphi(g \cdot x) = g \cdot \varphi(x).$$

which is to say, the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ g \downarrow & & \downarrow g \\ M & \xrightarrow{\varphi} & N \end{array}$$

Moreover, φ sends the orbit of x to the orbit of $\varphi(x)$.

Remark: If φ is an equivariant map, $\text{Stab } x \subset \text{Stab } \varphi(x)$.

For a fixed point $x \in M$, define an orbit mapping.

$$\begin{aligned} f_x: G &\longrightarrow M \\ g &\longmapsto g \cdot x \end{aligned}$$

the image of f_x is the $\text{Orb}(x)$, consider the differential map at $1 \in G$:

$$(df_x)_1: \mathfrak{g} \longrightarrow T_x M.$$

For each vector in Lie algebra $X \in \mathfrak{g}$, it induces a vector field $\tilde{X}$ on M :

$$\tilde{X}(x) := (df_x)_\mathbb{1} X.$$

this is called the **fundamental vector field associated with X** .

Remark: By our definition, the flow induced by $\tilde{X}(x)$ is: $\exp tX \cdot x$. The proof is simple:

Recall that a flow induced by $\tilde{X}(x)$ is a curve $\gamma(t)$ s.t.:

$$\tilde{X} \circ \gamma = \gamma'.$$

check that:

$$\begin{aligned} \tilde{X} \circ \gamma(t) &= (df_{\exp tX \cdot x})_\mathbb{1} X \\ &= \frac{df_{\exp tX \cdot x} \circ \exp sX}{ds} \Big|_{s=0} \\ &= \frac{d \exp sX \exp tX \cdot x}{ds} \Big|_{s=0} \\ &= \frac{d \exp(s+t)X \cdot x}{ds} \Big|_{s=0} \\ &= \frac{d \exp mX \cdot x}{dm} \Big|_{m=t} = \gamma'(t). \end{aligned}$$

Next, we consider the homogenous space $G/\text{Stab } x$, f_x induced a natural map

$$\begin{aligned} F_x: G/\text{Stab } x &\longrightarrow M \\ [g] &\longmapsto gx \end{aligned}$$

it is well-defined, since $[g'] = [g]$ if and only if $g'x = gx$.

Thm 1: $\bar{f}_x: G/\text{Stab } x \longrightarrow M$ is an injective immersion,

Proof: consider the tangent map of f_x at $1 \in G$:

$$(df_x)_1: \mathfrak{g} \longrightarrow T_x M.$$

the kernel of $(df_x)_1$ is the set:

$$\{X \in \mathfrak{g} : \tilde{X} \equiv 0\}.$$

which implies, if $X \in \ker(df_x)_1$, we have:

$$\frac{d \exp t X \cdot x}{dt} = 0 \quad \text{for all } t,$$

$\Rightarrow \exp t X \cdot x$ is a constant curve, and

$$\exp t X \cdot x = \exp 0 X \cdot x = x$$

$\Rightarrow X$ is in $L(\text{Stab } x)$.

By the commutative diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{(df_x)_1} & T_x M \\ (d\pi)_1 \downarrow & & \searrow \\ T_{[1]}(G/\text{Stab } x) & \xrightarrow{(d\bar{f}_x)_1} & T_x M \end{array}$$

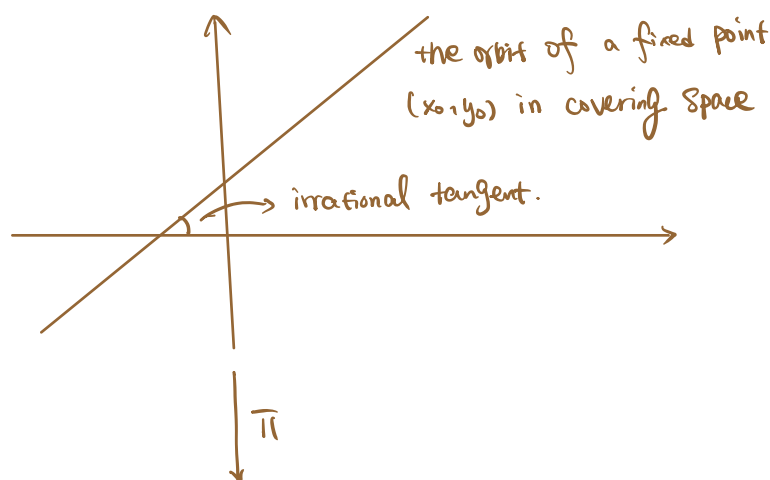
$$\text{we have: } \ker(d\bar{f}_x)_1 = (d\pi)_1 \ker(df_x)_1 = (d\pi)_1 L(\text{Stab } x) = 0$$

thus it is an immersion.

Remark: Although the image of f_x is the orbit of x $\text{orb}(x)$, but $\text{orb}(x)$ may be not a submanifold of M , for example: let $GL_1(\mathbb{R}) = \mathbb{R}^*$ acts on $S^1 \times S^1$

$$t \cdot (x, y) \longmapsto (x + t, x + \alpha t)$$

for some fixed $\alpha \in \mathbb{Q}^c$.



For the topology exhibited from $F_x(\mathbb{R}^*/\text{Stab}_x)$ is not same as the original one in T^2 .

Thm 2 : Any orbit of a compact Lie group action is a submanifold.

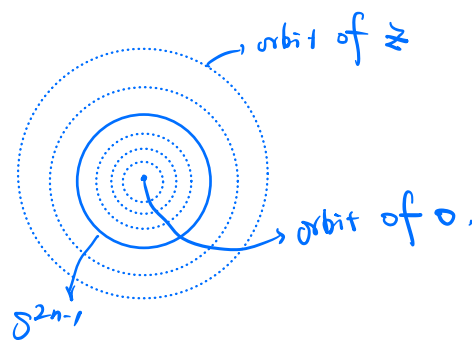
Proof : Recall that a proper injective immersion is an embedding, thus just show that F_x is in this case a proper one.

$$\begin{array}{ccc} G & \xrightarrow{f_x} & \\ \pi \downarrow & & \searrow \\ G/\text{Stab } x & \xrightarrow{\bar{f}_x} & M \end{array}$$

For a compact $K \subset M$, $\bar{f}_x^{-1}(K) = \pi \circ f_x^{-1}(K)$, } thus compact
 Since f_x is continuous, $f_x^{-1}(K)$ is closed in G , since π is a closed map, $\pi(f_x^{-1}(K))$ is compact

Examples : 1. S^1 acts on $\mathbb{C}^n$ by complex multiplication:

$$e^{i\theta} \cdot (z_1, \dots, z_n) := (e^{i\theta} z_1, \dots, e^{i\theta} z_n).$$



Except 0 as the fixed point, all the other orbits are circles, the sphere S^{2n-1} is stable under this action,

An G -action is said to be **effective**, if each element moves at least one x in M , i.e. $\forall g \in G, \exists x \in M$ st: $gx \neq x$, also equivalent to;

$$\bigcap_{x \in M} \text{Stab } x = \{1\}.$$

(For if $g \in \bigcap_{x \in M} \text{Stab } x \Leftrightarrow gx = x, \forall x \in M \Leftrightarrow G$ -action is not effective, unless $g=1$.)

Proposition: The group $\bigcap_{x \in M} \text{Stab } x$ is a closed normal subgroup in G , G -action on M will induce an effective action of $G / \bigcap_{x \in M} \text{Stab } x$ on M .

Proof: $\bigcap_{x \in M} \text{Stab } x$ is clearly a Lie subgroup of G , for $g \in \bigcap_{x \in M} \text{Stab } x, \forall h \in G$,

for any x ,

$$h^{-1}ghx = h^{-1}g(hx) = h^{-1}hx = x$$

$$\Rightarrow h^{-1}gh \in \bigcap_{x \in M} \text{Stab } x, \text{ thus a normal subgroup.}$$

Let: $[g] \cdot x := gx$, it clearly well-defined, and effective.

Example: 1. S^1 acts on $S^3 \subset \mathbb{C}^2$ via:

$$t(z_1, z_2) := (t^{m_1} z_1, t^{m_2} z_2), \quad m_1, m_2 \in \mathbb{Z}.$$

It is effective, iff $(m_1, m_2) = 1$

Proof: If for some $t \in S^1$ s.t.:

$$(t^{m_1} z_1, t^{m_2} z_2) = (z_1, z_2)$$

for all $z_1, z_2 \in S^3$, which if and only if:

$$t^{m_1} = t^{m_2} = 1 = e^{2k\pi i}$$

$$\text{thus } t = e^{\frac{2k\pi i}{m_1}} = e^{\frac{2n\pi i}{m_2}} \quad \text{where } k=1, \dots, m_1-1, n=1, \dots, m_2-1$$

$$\Rightarrow k = n \frac{m_1}{m_2}.$$

WLOG, let $m_1 < m_2$, since $m_2 \nmid n \Rightarrow m_1$ must contain a factor of m_2 , i.e. $(m_1, m_2) \neq 1$.

2. If G is simple, all the action is effective, or trivial, such as SO_3 .

The last thing is about the orbit space.

Define an equivalence relation on M :

$$x \sim_G y \iff x, y \text{ are in a same } G\text{-orbit,}$$

the quotient space is denoted by M/G , the orbit space.

The quotient space may be even not a Hausdorff space,

Example: $\mathbb{C}^* = GL_1(\mathbb{C})$ act by multiplication on $\mathbb{C}^{n+1}$,

$$\lambda \cdot z = \lambda z.$$

then the only nbhd containing $[0]$ is $\mathbb{C}^{n+1}/\mathbb{C}^*$, thus not a

Hausdorff one.

Proof: The orbit of $\mathbb{C}^*$ is the set:

$$\{\lambda z : \lambda \in \mathbb{C}^*\},$$

thus a nbhd of $[0]$ is the collection:

$$U = \left\{ \{\lambda z : \lambda \in \mathbb{C}^*\} : z \in U_0 \right\}$$

where U_0 is the nbhd of 0 in $\mathbb{C}^{n+1}$, consider the $\pi^{-1}(U)$

$$\pi^{-1}(U) = \bigcup_{z \in U_0} \{\lambda z : \lambda \in \mathbb{C}^*\} \subset \mathbb{C}^{n+1},$$

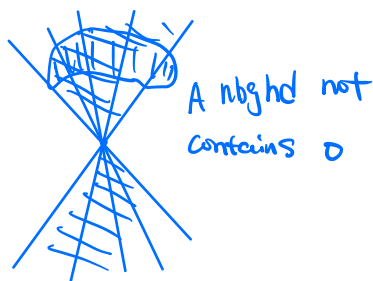
for any $z_0 \in \mathbb{C}^{n+1} \setminus \{0\}$, there exists an $z \neq 0$ in U_0 , such that:

$$\lambda = \frac{z_0}{z} \in \mathbb{C}^*$$

$$\Rightarrow z_0 \in \pi^{-1}(U) \Rightarrow \pi^{-1}(U) = \mathbb{C}^{n+1} \Rightarrow U = \mathbb{C}^{n+1} / \mathbb{C}^*,$$

thus not a Hausdorff Space.

However $(\mathbb{C}^{n+1} / \mathbb{C}^*) \setminus \{[0]\}$ is the Hausdorff one, which is $\mathbb{C}P^n$,



Thm: If G is a compact Lie group, M/G is Hausdorff, the Quotient map $\pi: M \rightarrow M/G$ is a proper and closed mapping.

Proof: First, π is an open map, for if U is open in M , to check if $\pi(U)$ an open set, just check that:

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g \cdot U$$

it is a union of open sets, thus $\pi^{-1}(\pi(U))$ is open.

If G is compact, π is also closed, for any closed subset $A \subseteq M$, $\pi(A)$ is closed iff $\pi^{-1}(\pi(A))$ is closed,

$$\pi^{-1}(\pi(A)) = \Theta(G \times A)$$

where Θ is the action map:

$$\Theta: G \times M \rightarrow M.$$

thus it suffices to show Θ is a closed map, whenever G is compact.

Let $C \subseteq G \times M$ be any closed subset, $y \in \overline{\Theta(C)}$, then $\exists$ sequence $\{g_n\}, \{x_n\}$ such that:

$$y = \lim_n g_n \cdot x_n.$$

Since G is compact, thus $g_n \rightarrow g \in G$, thus:

$$x_n = g_n^{-1} \cdot g_n \cdot x_n \rightarrow g^{-1}(y)$$

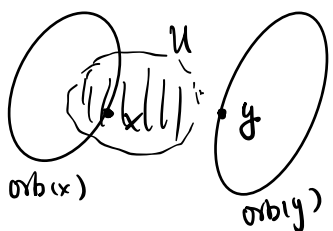
thus: $(g_n, x_n) \rightarrow (g, g^{-1}(y)) \in C$, thus:

$$y = g \cdot g^{-1}(y) \in \Theta(C) \Rightarrow \pi \text{ is closed.}$$

Since π is closed, fibers of π are compact, hence π is proper.

Finally, to show M/G is a Hausdorff space;

Let $\text{orb}(x), \text{orb}(y)$ be two distinct orbits, they're compact, since M is Hausdorff, $\exists U$, a neighborhood of x , st: $\bar{U} \cap \text{orb}(y) = \emptyset$. then



$\pi(U), \pi(\bar{U})$ are both open, they are not intersect, contains $\pi(x), \pi(y)$.

2. Equivariant Tubular Neighborhood Theorem

Now, we consider the Compact Lie Group action $G \times M \rightarrow M$, in this case the orbit $\text{orb}(x)$ is an embedded submanifold of M .

The usual tubular nbghd Thm state the relation between the original manifold and the normal bundle;

If $N \hookrightarrow M$ is the embedded submanifold of M , for each $x \in N$, define the normal space:

$$N_x := T_x M / T_x N$$

Then:

$$\mu_N := \bigsqcup_{x \in N} N_x$$

is the normal bundle of N , then N will be the embedded submanifold of μ_N via the zero section:

$$N \xrightarrow{i_0} \mu_N$$

$$x \mapsto (x, 0)$$

then there exists a nbghd $\underset{U}{\uparrow}$ of N in μ_N and a nbghd $\underset{V}{\uparrow}$ of N in M and

a diffeomorphism $\varphi: U \rightarrow V$ st the diagram commutes:

$$\begin{array}{ccc}
 N & \xrightarrow{i_0} & U \subset \mu_N \\
 & \searrow i & \swarrow \varphi \\
 & & V \subset M
 \end{array}$$

Here, we take $N = \text{orb}(x)$, Let $V_x = T_x M / T_x \text{orb}(x)$ be the normal space.
The normal bundle is endowed with a G -action;

$$\begin{aligned}
 G \times \text{orb}(x) \times V_x &\longrightarrow \text{orb}(x) \times V_x \\
 (g, (y, [X])) &\longmapsto (gy, [dg]_x \cdot X)
 \end{aligned}$$

Now, we wonder whether exists an equivariant diffeomorphism φ from some equivariant tubular nbhd to another.

Thm: There exists an equivariant nbhd of $\text{orb}(x)$ in $\text{orb}(x) \times V_x$, a
nbhd U of $\text{orb}(x)$ in M , and an equivariant diffeomorphism:

$$\varphi: U \longrightarrow V \text{ s.t.}$$

$$\begin{array}{ccc}
 \text{orb}(x) & \xrightarrow{i_0} & U \subset \text{orb}(x) \times V_x \\
 & \searrow i & \swarrow \varphi \\
 & & V \subset M
 \end{array}$$

commutes.

Proof: **Lemma 1:** If M equipped with a G -invariant metric, then, the exponential map $\exp: W \subset T_x M \longrightarrow \exp(W) \subset M$ is an equivariant map.

Proof: All what we need to prove is the following diagram commutes:

$$\begin{array}{ccc} T_x M & \xrightarrow{(dg)_x} & T_{g_x} M \\ \exp \downarrow & & \downarrow \exp \\ M & \xrightarrow{g} & M \end{array}$$

or just:

$$g \circ \exp tX = \exp (dg)_x tX$$

First notice that, since $\exp tX$ is a geodesic, g is an isometry thus LHS is a geodesic,

$$\text{RHS} = \exp t(dg)_x X$$

which is also a geodesic, pass the initial point g_x , with initial velocity $(dg_x)X$, now, let $\gamma(t) = g \circ \exp tX$,

$$\gamma(0) = g \circ \exp 0 = g_x,$$

$$\begin{aligned} \gamma'(0) &= \left. \frac{d g \circ \exp tX}{dt} \right|_{t=0} = (dg)_x \left(\left. \frac{d \exp tX}{dt} \right|_{t=0} \right) \\ &= dg_x X \end{aligned}$$

According to the uniqueness of geodesic, the lemma is proved.

Now, we wish to equip a G -invariant metric on M , then, we can identify the normal space V_x with the orthogonal complement: $T_x \text{orb}(x)^\perp$, since the exponential map is a diffeomorphism from a nbhd of M in TM to M :

$$\exp: W \subset TM \longrightarrow M$$

we can choose $U = W \cap \mu_{\text{orb}(x)}$, which is a nbhd of M in $\mu_{\text{orb}(x)}$ and:

$$\exp|_U: U \subset \mu_{\text{orb}(x)} \longrightarrow \exp(U)$$

is an equivariant diffeomorphism by the previous lemma, so we just need to prove the existence of the G -invariant metric.

Lemma: If G is a compact Lie group acts on M , then there exists a G -invariant metric on M .

Proof: choose $\omega \in \Omega^n(G)$, where $n = \dim G$, let:

$$\omega_G = g \cdot \omega, \quad g \in G.$$

it is a left invariant n -form on G , define:

$$|G| = \int_G \omega_G.$$

for any metric $\langle \cdot, \cdot \rangle$ defined on M , define:

$$\{x, Y\} := \frac{1}{|G|} \int_G \langle dg_x x, dg_x Y \rangle \omega_G.$$

it is G -invariant, for any $h \in G$:

$$\begin{aligned} \{dh_x x, dh_x Y\} &= \frac{1}{|G|} \int_G \langle dg_{hx} dh_x x, dg_{hx} dh_x Y \rangle \omega_G \\ &= \frac{1}{|G|} \int_G \langle dg_{hx} dh_x x, dg_{hx} dh_x Y \rangle L_{h^{-1}}^* \omega_G \\ &= \frac{1}{|G|} \int_G \langle dg_x x, dg_x Y \rangle \omega_G \\ &= \{x, Y\}. \end{aligned}$$

That ends the proof.

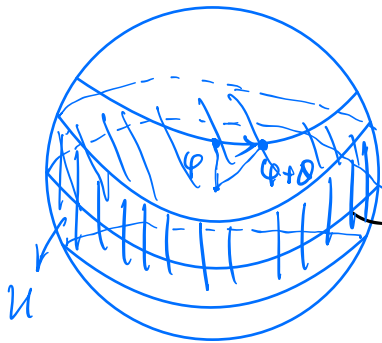
Example: S^1 acts on S^2 by rotation:

$$\theta: (\varphi, h) \mapsto (\varphi + \theta, h).$$

Then every orbit is the ~~well~~, choose the equator

circle as the orbit of $(\varphi, 0) := x$, then the

tubular nbhd of Gx is the whole normal



Gx bundle: $S^1 \times \mathbb{R}$ — a cylinder, a tubular of

Gx in S^2 is the set:

$$U = \left\{ (\varphi, h) \mid \varphi \in [0, 2\pi], |h| < \frac{1}{2} \right\}.$$

the equivariant diffeomorphism:

$$\varphi: S^1 \times \mathbb{R} \longrightarrow U$$

$$(\theta, x) \longmapsto \left(\theta, \frac{x}{2\sqrt{1+x^2}} \right)$$

Lecture 2: Some Applications.

Remarks: Using above notation, for any $z \in M$, we can find a $g \in G_z$, thus we have an isomorphism in the level of tangent space:

$$dg_z : T_z M \longrightarrow T_z M.$$

Moreover, since g sends the orbit $G \cdot x$ to $G \cdot z$, thus $T_z G \cdot x$ is an invariant subspace of dg_z . thus it will induce a linear representation of G_z (the isotropy group):

$$\begin{aligned} \phi : G_z &\longrightarrow GL(V_z), \\ g &\longmapsto \tilde{dg}_z : [v] \longmapsto [dg_z v]. \end{aligned}$$

which is also an action on V_z , thus G_z has a natural action on $G \cdot V_z$, via:

$$h \cdot (g, [v]) = (gh^{-1}, [\tilde{d}h_x v])$$

this action is obviously free, thus the orbit space $(G \cdot V_z) / G_z$ is a manifold, which is isomorphic to the normal bundle of orbit $G \cdot x$, and it is endowed with a G -action: $\forall g' \in G$:

$$g' \cdot [g, [v]] = [g'g, [v]].$$

(Here is a problem, the normal bundle, $\text{Nor}(G \cdot x)$, which we mentioned last week, is endowed with a natural action of G :

$$g \cdot (x, [X]) = (g \cdot x, [dg_x X])$$

the isomorphism between $\text{Nor}(G \cdot x)$ and $(G \cdot V_x) / G_x$ is as follows:

$$\phi([g, [v]]) = [g \cdot x, [v]]$$

I want to see if the action we mentioned above on $(G \cdot V_x) / G_x$ is the same as the one in $\text{Nor}(G \cdot x)$, i.e., the following diagram commutes:

$$\begin{array}{ccc} (G \cdot V_x) / G_x & \xrightarrow{\phi} & \text{Nor}(G \cdot x) \\ g' \downarrow & & \downarrow g' \\ (G \cdot V_x) / G_x & \xrightarrow{\phi} & \text{Nor}(G \cdot x) \end{array}$$

but shamefully, I didn't figure it out.)

Thus in later, we shall use the notation $(G \times V_x)/G_x$ to replace for the normal bundle of $\text{orb}(x)$.

Theorem 1 : The union of all orbits of a given type is a submanifold.

Remark: In previous Lecture, we showed that, each orbit $G \cdot x$ will induce a type of conjugations namely (G_x) , this correspondence is not injective, i.e., some different orbit $G \cdot x \neq G \cdot y$ can induce the same orbit type:

$$(G_x) = (G_y).$$

For example in the S^1 -action on S^2 by rotation, any two orbit except the two poles has the same orbit type, for we can find two different points in different orbits with different latitude but same longitude, they have the same stabilizer.

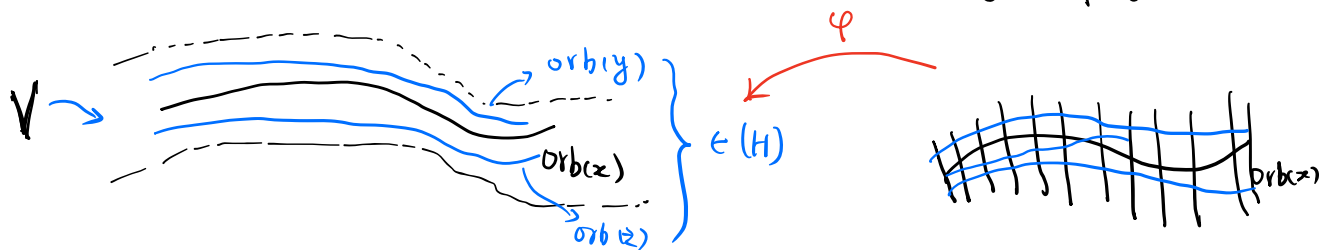


Proof: Fix a type (H) , where H is the Lie subgroup of G , the union of all the orbits of type (H) is:

$$M_{(H)} = \{ x \in M \mid (G_x) = (H) \}.$$

To show $M_{(H)}$ is a submanifold of M , it suffices to show it is a submanifold in a nbhd of an orbit $\text{orb}(x)$,

Now, we choose a equivariant tubular nbhd V of $\text{orb}(x)$, the existence of which is proved last week, then V must contain some entire orbit of $M_{(H)}$, i.e they're of type (H) .



by the equivariant diffeomorphism φ , the pre-image of $V \cap M_{(H)}$ are some orbits in the normal bundle $(G \times V_x)/G_x$, which are also of type (H),

Now, we shall consider the union of orbits of G -action on $(G \times V_x)/G_x$ with type (H), is it a subbundle of the normal bundle? $\left((G \times V_x)/G_x \right)_{(H)}$.

if $g' \in G[g, v]_I$, which implies: $g' \cdot [g, v] = [g'g, v] = [g, v]$

which if and only if: $\exists h \in G_x$, s.t. $g'g = gh^{-1}$, $v = h \cdot v$

if and only if: $\exists h \in (G_x)_v$, s.t. $g' = gh^{-1}g^{-1}$

if and only if: $g' \in (G_x)_v$

thus an orbit type of $(G \times V_x)/G_x$ is of (H) if and only if:

$$[g, v] \in (G_x)_v \in (H) = (G_x) \Rightarrow (G_x)_v = G_x$$

which if and only if v is a fixed point of G_x action on V_x .

$$\text{Let } F = \{ v \in V_x \mid hv = v, \forall h \in G_x \}$$

it is a subspace of V_x , and moreover:

$$\begin{aligned} \left((G \times V_x)/G_x \right)_{(H)} &= \{ [g, v] \in (G \times V_x)/G_x; v \in F \} \\ &= (G \times F)/G_x \end{aligned}$$

which is a subbundle of $(G \times V_x)/G_x$, thus $M_{(H)}$ is a submanifold of M .

Now, if take (H) to be (G), the union of all orbits of type (G) is the fixed points of G -action on M , denoted by M^G

Theorem 2: M^G is a submanifold of M .

If we suppose M is a compact manifold, G is the compact Lie group S^1 , then we have:

Theorem 3: $\chi(M) = \chi(M^S)$. (Kobayashi - Conner Theorem)

Aspect of the proof: Let V be a tubular nbhd of $M^{S'}$, $U = M \setminus \overline{M^{S'}}$.

Then Applying Myer-Vietoris theorem, we have:

$$\chi(M) = \chi(U) + \chi(V) - \chi(U \cap V).$$

Since U can contract to $M^{S'}$, thus it suffices to show $\chi(V) = \chi(U \cap V) = 0$.

Since any subgroup $\mathbb{Z}_p \leq S'$ with p prime acts semi-freely on M , (because any nontrivial element of $\mathbb{Z}_p$ is a generator), we have:

$$M^{S'} = M^{\mathbb{Z}_p}$$

(For $M^{S'} \subset M^{\mathbb{Z}_p}$, if $x \in M^{\mathbb{Z}_p}$, $\forall m \in \mathbb{Z}_p$, $m \cdot x = x$, we want to show $\forall g \in S'$, $g \cdot x = x$,

in fact, since $\mathbb{Z}_p$ acts semi-freely, $\mathbb{Z}_p \leq S'_x \leq S'$, since all subgroup of S' has the form $\mathbb{Z}_m$ (which will be proved by W. Shen in the next course, $\mathbb{Z}_p = S'_x$, thus,

$(S'_x g x) = (\mathbb{Z}_p)$, i.e. $g x = g' m g'^{-1} x$ for some $g' \in S'$, since S' is Abelian,

$$g x = g' g'^{-1} m x = x \Rightarrow M^{S'} = M^{\mathbb{Z}_p} \quad)$$

and $\mathbb{Z}_p$ acts on V freely, thus: (the same as on $U \cap V$)

$$\chi(V) = p \cdot (V/\mathbb{Z}_p) \text{ for all } p.$$

$$\Rightarrow \chi(V) = 0 = \chi(U \cap V).$$

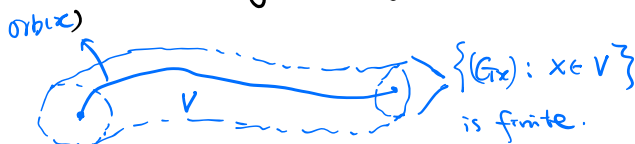
Theorem 4: If M is compact, there are only finitely many orbit types:

Proof: We'll use the induction on $\dim M = n$.

if $\dim M = 0$, M itself is a isolated finitely set, easy.

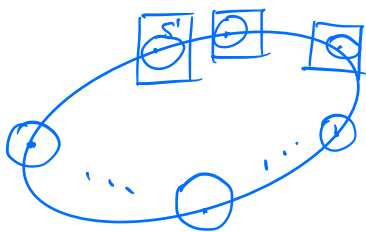
Suppose the result holds up to $\dim M = n-1$, for $\dim M = n$, since M is compact, it suffices to show that M has only finitely many orbit types in an nbhd of $\text{orb}(x)$.

equivariant tubular



To see this, we took a glance in the level of normal bundle, $(G \times V_x)/G_x$.

We take a G_x -invariant metric on V_x , take the sphere bundle $(G \times SV_x)/G_x$, it is a compact manifold of dimension $n-1$, thus by the induction assumption, there is only finitely many orbit types.



Now, we concentrate on the orbit in $(G \times V_x)/G_x$. in fact we have:

$$(G[g, v]) = (G[g, \frac{v}{\|v\|}])$$

thus $(G \times V_x)/G_x$ also has only finitely many orbit types.

this same thing happens on M .
is connected.

Theorem 5: If M/G is connected, there is an orbit type (H) in M such that $M_{(H)}$ is a dense open subset of M (compact), moreover $M_{(H)}/G$ is a connected manifold.

Such orbits i.e the $G \times x$ with $\overline{M_{Gx}} = M$ are called principal orbits. If all the orbits are principal, the action is said to be principal. (G is connected compact Lie group)

Proof: We use the induction on $\dim M = n$, if $\dim M = 0$, M/G is connected, there is only one orbit in M , obviously: $M_{(H)} = M$.

Suppose the result holds up to $\dim M \leq n-1$, for $\dim M = n$, we'll show that the principal orbit exists in a normal bundle $(G \times V_x)/G_x$.

Again, we wish to use the sphere bundle $(G \times SV_x)/G_x$ to apply the induction assumption, however, we need to show $((G \times SV_x)/G_x)/G := SN/G$ is connected, it suffices to prove SN is connected.

Let $\pi: SN \rightarrow G/G_x$ be the bundle projection, for $a, b \in SN$, choose a path which joining $\pi(a), \pi(b)$, (The orbit is connected), then lift it to the SN .

$$\begin{array}{ccc}
 & \nearrow \tilde{\rho} & SN \\
 [0,1] & \xrightarrow{\rho} & G/G_x
 \end{array}$$

thus the principal orbit exists for SN/G , again:

$$(G[g, vI]) = (G[g, \frac{v}{\|v\|}I])$$

the same thing will happen in N , and M .

Remarks: 1. If G is a connected commutative Lie group (i.e. $G \cong T^n \cong S^1 \times S^1 \times \dots \times S^1$),
Abelian compact

let (H) be the type of the principal orbit, then every point x in M is a fixed point of H ,

(Proof): Since $\overline{M_{(H)}} = M$, thus $\forall x \in M, \exists x_n \in M_{(H)}$ st: $x_n \rightarrow x$,

then for each x_n , $G_{x_n} \in (H)$ or $H \in (G_{x_n})$, thus $\forall h \in H, \exists g_n \in G_{x_n}$, $g'_n \in G$, st: $h = g'_n g_n g_n^{-1}$, then:

$$h \cdot x_n = g'_n g_n g_n^{-1} x_n = g'_n g_n^{-1} g_n x_n = x_n$$

by letting $n \rightarrow \infty$, $hx = x$.

Thus if the acting is effective, we have

$$H \subset \bigcap_{x \in M} G_x = \{1\},$$

2. An orbit is principal, if and only if the representation in V_x of its stabiliser is trivial.

Lecture N: Properties of the Moment Maps

N.1 The fixed points of the group action.

Theorem 1: Let M be a compact symplectic manifold endowed with an S^1 -action, and if it is Hamiltonian then it has fixed points.

Before I give the proof, we should know the notions about the infinitesimal view of group action: $G \times M \rightarrow M$. Since a Lie group G is a "continuous" transformation group, thus, it allows us to think two elements $g_1, g_2 \in G$ to be "close enough", since the action is smooth, it leads the results $g_1 \cdot x, g_2 \cdot x$ will be so close in the level of M , then, translate it to the language in manifolds, the Lie algebra $\mathfrak{g}$ will act on M resulted in TM , i.e.:

$$\begin{aligned} \mathfrak{g} \times M &\longrightarrow TM \\ (X, m) &\longmapsto \underline{X}(m) \end{aligned}$$

this is the infinitesimal view of the Lie group actions, moreover, one has the following:

Lemma: $x \in M$ is a fixed point of the Lie group action $G \times M \rightarrow M$, if and only if $\underline{X}(x) = 0$ for all $X \in \mathfrak{g}$, in the infinitesimal view.

Proof: $\Rightarrow$: $\underline{X}(x) = \left. \frac{d(\exp tX)(x)}{dt} \right|_{t=0} = 0$, $\forall X \in \mathfrak{g}$, which is obviously.

$\Leftarrow$: Since $\underline{X}(x) = 0$, then $(\exp tX)(x) = x$ for all $t \in \mathbb{R}$, by the previously lemma:

$$\exp tX = \exp tX$$

Hence we have $(\exp tX)(x) = x$ for all $t \in \mathbb{R}$ and all $X \in \mathfrak{g}$, which implies, G doesn't move $x \in M$.

Proof of the thm 1: The moment map of $S^1 \times M \rightarrow M$ is just a function:

$$\mu: M \rightarrow \mathbb{R} \cong \text{Lie}(S^1)^*$$

Since M is compact, μ must have a critical point, saying $x \in M$, s.t.: $(d\mu)_x = 0$. which implies $\forall X \in \mathbb{R} \cong \text{Lie}(S^1)$, we have:

$$\left. \begin{aligned} \langle (d\mu)_x^\circ, X \rangle &= \omega(\cdot, \underline{X}(x)) \\ \parallel \\ X \cdot (d\mu)_x^\circ &= 0 \end{aligned} \right\} \Rightarrow \underline{X}(x) = 0, \text{ due to the non-degeneracy of } \omega, \text{ the symplectic form.}$$

that implies x is a fixed point due to the last lemma.

Remark 1: It is not a coincidence, in the next chapter, we will associate a function to the Hamiltonian torus action: $S^1 \times \dots \times S^1 \times M \rightarrow M$, such that the critical points of which correspond to the fixed point of the torus action, and which will allow one to use the method of Morse theory.

2): If the S^1 -action is just symplectic, then, it may have no fixed point:

Example: The rotation of one factor S^1 on a torus $T^2 \cong S^1 \times S^1$,

$$\begin{aligned} S^1 \times T^2 &\longrightarrow T^2 \\ t \cdot (x, y) &\longmapsto (tx, y) \end{aligned}$$



For any $X \in \mathbb{R} \cong \text{Lie}(S^1)$, $\exp tX$ on S^1 is the trajectory of a particle doing the uniform circular motion with the angular velocity X (rad/s), then the curve $(\exp tX)(x)$ on T^2 for $\forall x \in T^2$, is exactly a meridian, hence the fundamental vector field of X is:

$$\underline{X}_x(x) = X \cdot \frac{\partial}{\partial \theta}$$

which expressed in a local chart near x , with the symplectic form $\omega = dx \wedge dp$, one finds:

$$i_{\underline{X}} \omega = X \cdot dp.$$

However, we cannot deduce it is a Hamiltonian action, because $X \cdot dp$ is just defined locally not globally. There're two ways to see it is not Hamiltonian:

1: Consider the universal covering space of T^2 , the $\mathbb{R}^2$:

$$\begin{aligned} q: \mathbb{R}^2 &\longrightarrow T^2 \cong \mathbb{R}^2 / \mathbb{Z}^2 \\ (x, y) &\longmapsto (e^{2\pi i x}, e^{2\pi i y}) \end{aligned}$$

If $\mu = df$ is exact in T^2 , then $q^* \mu = df \circ q$ is exact on $\mathbb{R}^2$, thus:

$$q^* i_{\underline{X}} \omega = i_{q^* \underline{X}} q^* \omega = i_{X \frac{\partial}{\partial y}} \left(\frac{1}{4\pi} e^{2\pi i(x+y)} dx \wedge dy \right) = \frac{1}{4\pi} e^{2\pi i(x+y)} dx$$

which is obviously not exact, hence the action is not Hamiltonian.

2: Note that on a compact manifold, all Hamiltonian vector fields have a zero, for

$$X = X_f = J \text{grad} f$$

f will have a critical point on a compact space M , and since $\underline{X} = X \cdot \frac{\partial}{\partial \theta}$ is nowhere vanishing, thus, it is not a Hamiltonian action.

Back to the example, it obviously has no critical points.

N.2. The tangent Map.

Recall that in the previously classes, we claimed that:

$$\langle (d\mu)_x Y, X \rangle = -i_X \omega(Y)$$

Because one can regard $\mu: M \rightarrow \mathfrak{g}^*$ as the dual map of $\tilde{\mu}: \mathfrak{g} \rightarrow C^\infty(M) (= \mathfrak{m}^*)$

then $\langle \mu(x), X \rangle = \langle \tilde{\mu}(X), x \rangle$, hence $\langle (d\mu)_x Y, X \rangle = (d\tilde{\mu}_x)^T Y = (d\mu^*)_x Y = -i_X \omega(Y)$.

Now, if we regard $i_X \omega$ as a cotangent vector in $T_x^* M$, then we may write it as:

$$\langle (d\mu)_x Y, X \rangle = -\langle i_X \omega, Y \rangle$$

thus the tangent map of the moment map:

$$(d\mu)_x: T_x M \longrightarrow \mathfrak{g}^*$$

is the dual map (adjoint) of the map:

$$\begin{aligned} (d\mu)_x^T: \mathfrak{g} &\longrightarrow T_x^* M \\ x &\longmapsto i_X \omega \end{aligned}$$

Hence due to the well-known result in linear algebra:

$$\text{Im}(d\mu)_x = \left(\ker(d\mu)_x^T \right)^\circ$$

$$\Downarrow \\ (\text{Im}(d\mu)_x)^\circ = \ker(d\mu)_x^T$$

where $A^\circ \subset W^*$ represents for the annihilator for $A \subset V$.

However, notice that:

$$\begin{aligned} \ker(d\mu)_x^T &= \{ X \in \mathfrak{g} : (i_X \omega)_x = 0 \} \\ &= \{ X \in \mathfrak{g} : X \cdot x = 0 \} \\ &= \mathfrak{g}_x = \text{Lie}(G_x) \end{aligned}$$

Theorem 2: The moment map $\mu: M \rightarrow \mathfrak{g}^*$ is a submersion if and only if G_x is a discrete subgroup of G .

Recall from the 3rd class, the orbit type of a principal orbit is $\{1\}$ if G is the torus action, so:

Corollary: The moment map of a torus action is a submersion on all principal orbits.

Theorem 3: $\ker(d\mu)_x$ is symplectic orthogonal to $T_x G \cdot x$.

Proof: For any $Y \in \ker(d\mu)_x \subseteq T_x M$, $(d\mu)_x Y = 0$, which implies:

$$\langle (d\mu)_x Y, X \rangle = 0 = \omega(X, Y) \text{ for all } X \in \mathfrak{a}_g,$$

$\Rightarrow Y$ is symplectic orthogonal to $\{X(x) \mid X \in \mathfrak{a}_g\} = T_x G \cdot x$

N.3. Noether Theorem.

Theorem 4: $H: M \rightarrow \mathbb{R}$ invariant under G -action, then μ is constant along the trajectory of X_H .

Proof: $X_H \circ \gamma(t) = \gamma'(t)$, just to show that $\frac{d\mu \circ \gamma(t)}{dt} = 0$, which suffices to compute, $\forall x \in \mathfrak{a}_g$:

$$\frac{d}{dt} \langle \mu \circ \gamma(t), x \rangle = \left\langle \frac{d\mu \circ \gamma(t)}{dt}, x \right\rangle \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{chain rule}$$

$$= \left\langle (d\mu)_{\gamma(t)} X_H \gamma(t), x \right\rangle$$

$$= \left\langle X_H \circ \gamma(t), (d\mu)_{\gamma(t)}^T x \right\rangle$$

$$= \left\langle X_H \circ \gamma(t), -(\dot{\nu}_x \omega)_{\gamma(t)} \right\rangle$$

$$= -(\dot{\nu}_x \omega)_{\gamma(t)}(X_H \circ \gamma(t))$$

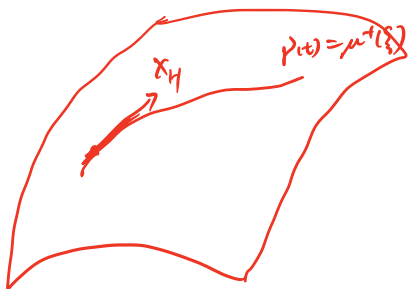
$$= -\omega(x, X_H \circ \gamma(t))$$

$$= (dH)_{\gamma(t)}(x)$$

Since H is invariant, i.e.

$$\text{differentiate at } s=0 \quad H(\exp s x) \cdot \gamma(t) = H(\gamma(t))$$

$$\Rightarrow (dH)_{\gamma(t)} = 0$$



Example: Here we take a naive example of $S^1 \times S^2 \rightarrow S^2$ by rotating. For any $X \in \text{Lie}(S^1) \cong \mathbb{R}$, X on S^2 is the tangent vector along the left, with module x , or write in the cylindrical coordinate is:

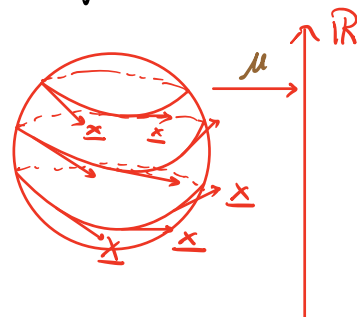
$$\underline{X}(\theta, h) = X \frac{\partial}{\partial \theta}$$

hence $\tilde{\mu}: X \mapsto (\theta, h) \mapsto X \cdot h$, thus the moment map

$$\begin{aligned} \text{is: } \mu: (\theta, h) &\mapsto (X \mapsto X \cdot h) \\ M &\longrightarrow \mathbb{R} \cong (\text{Lie}(S^1))^* \end{aligned}$$

which is exactly the Height function on S^2 !!!

And μ is obviously invariant under S^1 -action, X_μ is exact $\underline{X}$, which is tangent to $\mu^{-1}(\xi)$, $\forall \xi \in \mathbb{R}$, the flow of X_μ is exactly the left, μ is identical on the left. Moreover, the critical point of μ is exactly the fixed point of S^1 -action.



In the later chapter, those functions $H: M \rightarrow \mathbb{R}$ invariant under G , such that X_H lies in the tangent space of the level set $H^{-1}(x)$, will play an important role, such functions are called the periodic Hamiltonians.

N.4: Equivariant Moment Mappings

Theorem 5: Moment map associates to the connected Lie groups Hamiltonian Action is equivariant.

Before I give this proof, let's go back to the infinitesimal version of the "equivariant mapping"

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ g \downarrow & & \downarrow g \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

the infinitesimal sight of this diagram is:

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ m \mapsto \underline{X}(m) \downarrow & & \downarrow f(m) \mapsto \underline{X}(f(m)) \\ TM_1 & \xrightarrow{df} & TM_2 \end{array}$$

Where $\underline{x}^1, \underline{x}^2$ are the fundamental vector fields induced by $X \in \mathfrak{g}$ associated to M_1, M_2 respectively.
 a natural question is, are these two things mean the same?

However, when G is non-connected, they are not true, for example: $G = \mathbb{Z}_p$, then the infinitesimal equivariantness is a "Junk Word", however, in the case of connected Lie group, they are incredibly same.

Lemma: In the sense of connected Lie group G -action, the equivariant means equivariant infinitesimally.

Proof: $\Rightarrow$ is trivial.

$\Leftarrow$ Recall that $(\exp tX)^! \cdot m$ is exactly the flow of $\underline{x}^1$ on M_1 , then we claim f is equivariant at least for $\exp tX$, which means:

$f((\exp tX)^! \cdot m)$ is the flow of $\underline{x}^2$ (starting from $f(m)$)

$$\begin{aligned} \text{Indeed, } \underline{x}^2 \left(f((\exp tX)^! \cdot m) \right) &= (df)_{(\exp tX)^! \cdot m} \underline{x}^1 \circ (\exp tX)^! \cdot m \quad (\text{Infinitesimally Equivariant}) \\ &= (df)_{(\exp tX)^! \cdot m} \underline{x}^1(m) \\ &= \frac{d(f \circ \exp tX)_m}{dt} \end{aligned}$$

Hence f fixes with all elements such as $\exp tX$, whenever G is connected, G is actually generated by those $\exp tX$.

Proof of thm 5.

Just to compute in the infinitesimal case:

$$\begin{array}{ccc} M & \xrightarrow{\mu} & \mathfrak{g}^* \\ \downarrow x \mapsto \underline{x}^M(x) & & \downarrow \mu(x) \mapsto \underline{x}^{\mathfrak{g}^*}(\mu(x)) \\ TM & \xrightarrow{d\mu} & \mathfrak{g}^* \end{array}$$

$$(d\mu)_x \underline{x}^M(x) = \underline{x}^{\mathfrak{g}^*}(\mu(x)).$$

To see this, $\forall Y \in \mathfrak{g}$, on the one hand:

$$\begin{aligned} \left\langle \underline{x}_{(\mu(x))}^{g^*}, Y \right\rangle &= \left\langle \mu(x), [Y, x] \right\rangle \\ &= -\tilde{\mu}_{[x, Y]}^{(x)} \end{aligned}$$

On the other hand:

$$\begin{aligned} \left\langle (d\mu)_x \underline{x}^M(x), Y \right\rangle &= \left\langle \underline{x}^M(x), (d\mu)_x^T Y \right\rangle \\ &= \left\langle \underline{x}^M(x), i_Y^M \omega \right\rangle \\ &= \omega_x(\underline{x}^M, Y^M) \\ &= -\{\tilde{\mu}_x, \tilde{\mu}_Y\}^{(x)} \\ &= -\tilde{\mu}_{[x, Y]}^{(x)}. \end{aligned}$$

Since G is connected, that ends our proof.