Notes on Symplectic Geometry

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3 Symplectic And Hamiltonian Group Actions

3.1 Hamiltonian group actions

Fundamental vector field and infinitesimal action.

Definition 1. Suppose a Lie group G acts smoothly on M. For simplicity we always assume that G is connected.

$$\begin{aligned} \Phi : \ G \times M &\longrightarrow M \\ (g, x) &\longmapsto \Phi(g, x) = \Phi_g(x) \end{aligned}$$

Associated to each vector $X \in \mathfrak{g} = Lie(G)$, the map $\Phi^X : \mathbb{R} \times M \to M$, defined by

$$\Phi^X(t, x) = \Phi(\exp(tX), x),$$

is an $\mathbb{R}$-action on M. In other words, $\Phi_{\exp(tX)} : M \to M$ is a flow on M. The vector field X_M on M, given by

$$X_M := \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp(tX)}(x),$$

is called the **infinitesimal generator** of the action corresponding to X.

Remark 1. So there is a map

$$\beta : \mathfrak{g} \to \mathrm{Vect}(M), \quad X \mapsto X_M.$$

It is called the **infinitesimal action** of $\mathfrak{g}$, and this a Lie algebra anti-homomorphism, i.e.

$$[X, Y]_M = -[X_M, Y_M].$$

Proof. First, we will show a lemma

Lemma 1. $(\mathrm{Ad}_g X)_M = (\Phi_g)_* X_M$

□

This property can be used in the proof of the symplectic forms on the coadjoint orbit (see Exercise II.29 in [Aud04]).

One can define $\gamma(X) := -\beta(X)$, then $\gamma : \mathfrak{g} \to \mathrm{Vect}(M), \quad X \mapsto \underline{X} := -X_M$ is a Lie algebra homomorphism, where $\underline{X}$ is called the **fundamental vector field** in [Aud04].

3.1.1 Symplectic actions.

Now suppose M is symplectic with symplectic form ω. The G-action is called **symplectic** if any element g of G defines a diffeomorphism which preserves ω (symplectomorphism):

$$\Phi_g^* \omega = \omega.$$

Let us write the infinitesimal version of this equality, and then we can know that the fundamental vector field is symplectic (locally Hamiltonian).

Proposition 1. All the fundamental vector fields of a symplectic action are symplectic vector fields.

Proof. Let X $\in \mathfrak{g}$ and let X_M be the associated infinitesimal generator. Let ϕ_t be the flow of X_M,

$$\phi_t(x) = \exp(tX) \cdot x = \Phi_{\exp(tX)}(x).$$

Since for any element of G satisfies that $\Phi_g^* \omega = \omega$, that is for any $X \in \mathfrak{g}$, $\Phi_{\exp(tX)}^* \omega = \omega$.

$$L_{X_M}\omega = \frac{d}{dt}\Big|_{t=0} \Phi_{\exp(tX)}^* \omega = 0.$$

We use now both Cartan formula and the fact that ω is closed, to get:

$$d i_{X_M} \omega = 0.$$

□

Now, one can describe the Symplectic G-action by the fundamental vector field according to this proposition.

3.1.2 Hamiltonian actions.

Definition 2. Similarly, the Symplectic G-action is called

- **weakly Hamiltonian** G-action if every fundamental vector field $\underline{X}$ is Hamiltonian, with Hamiltonian functions $\tilde{\mu}_X$,that is $\underline{X} = X_{\tilde{\mu}_X}$,

$$i_{\underline{X}}\omega = -d\tilde{\mu}_X.$$

  where $\tilde{\mu} : \mathfrak{g} \to C^\infty(M),\ X \mapsto \tilde{\mu}_X$.

- **Hamiltonian** G-action if it is weakly Hamiltonian, and the map $\tilde{\mu}$ is a Lie algebra homomorphism. One says that the morphism $\tilde{\mu}$ of Lie algebra $\tilde{\mu} : (\mathfrak{g}, [\cdot,\cdot]) \to (C^\infty(M), \{\cdot,\cdot\})$ makes the following diagram commute.

$$\begin{array}{ccccccccc} & & C^\infty(M) & \xleftarrow{\ \tilde{\mu}\ } & \mathfrak{g} & & & & \\ & & \nu\downarrow & & \downarrow\gamma & & & & \\ 0 & \longrightarrow & \mathrm{Ham}(M) & \xrightarrow{\ i\ } & \mathrm{Symp}(M) & \longrightarrow & H^1(M;\mathbb{R}) & \longrightarrow & 0 \end{array}$$

  where the i denotes the inclusion, $\gamma := -\beta$, thus γ is a Lie algebra morphism, where β is the infinitesimal action of Lie algebra $\mathfrak{g}$ on M.[1]

Remark 2. What happens when we do not require that $\tilde{\mu}$ is a Lie algebra morphism ?

Suppose that $X \mapsto \tilde{\mu}_X$ is only a map from $\mathfrak{g}$ to $C^\infty(M)$.Then if we compute

$$\nu \circ \tilde{\mu}([X,Y]) = \gamma([X,Y]) = [\gamma(X), \gamma(Y)] = [\nu \circ \tilde{\mu}(X), \nu \circ \tilde{\mu}(Y)]$$
$$\nu(\tilde{\mu}_{[X,Y]}) = \nu(\{\tilde{\mu}(X), \tilde{\mu}(Y)\}) = \nu(\{\tilde{\mu}_X, \tilde{\mu}_Y\})$$

we note that the problem may be solved up to a constant. Define $c(X,Y) \in \mathbb{R}$ by

$$c(X,Y) = \{\tilde{\mu}_X, \tilde{\mu}_Y\} - \tilde{\mu}_{[X,Y]}.$$

Consequently, $\tilde{\mu}$ is a Lie algebra homomorphism implies that the constant is vanished.

[1]Since the action is Symplectic G-action, the fundamental vector field is indeed a Symplectic vector field.

3.1.3 The momentum map.

Definition 3. Associated with $\tilde{\mu}$ is the **momentum map**:

$$\begin{aligned} \mu : M &\longrightarrow \mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, \mathbb{R}) \\ x &\longmapsto (X \mapsto \tilde{\mu}_X(x)). \end{aligned}$$

where $\tilde{\mu}_X(x) =< \mu(x), X >$. Then the fundamental vector field $\underline{X}$ is the Hamiltonian vector field of the function $\tilde{\mu}_X$.

Remark 3. The condition that $\tilde{\mu}_X$ is Hamiltonian for the fundamental vector field $\underline{X}$ translates into a condition on the tangent map.

$$\begin{aligned} T_x\mu : T_xM &\longrightarrow \mathfrak{g}^* \qquad x \in M, Z \in T_xM \\ Z &\longmapsto T_x\mu(Z) \end{aligned}$$

namely, $\forall X \in \mathfrak{g}$,

$$d\tilde{\mu}_X(Z) =< T_x\mu(Z), X >= -i_{\underline{X}}\omega.$$

How to describe Hamiltonian action via the momentum map μ instead of $\tilde{\mu}$?

Definition 4. The momentum is **G-equivariant** or **Ad*-equivariant** if the follow diagram commute.

$$\begin{array}{ccc} M & \xrightarrow{\mu} & \mathfrak{g}^* \\ {\scriptstyle \Phi_g}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}_g^*} \\ M & \xrightarrow{\mu} & \mathfrak{g}^* \end{array}$$

where $g \in G$,

$$\mu \circ \Phi_g = \mathrm{Ad}_g^* \circ \mu.$$

Proposition 2. The momentum map μ is G-equivariant if and only if the map $\tilde{\mu} : (\mathfrak{g}, [\cdot, \cdot]) \to (C^\infty(M), \{\cdot, \cdot\})$ is a Lie algebra homomorphism.[2]

Proof. There is a proof on Page 203 of the book [MS17]. □

Proposition 3. Let $f : M \to N$ be an G-equivariant smooth map. Then for any $X \in \mathfrak{g}$ we have:

$$Tf \circ X_M = X_N \circ f.$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle X_M}\downarrow & & \downarrow{\scriptstyle X_N} \\ TM & \xrightarrow{Tf} & TN \end{array}$$

[2] We assumed that G is connected.

Proof. By equivariance we have:

$$f \circ \Phi_{exp(tX)} = \Psi_{exp(tX)} \circ f.$$

Differetiating with respect to t at $t = 0$ and using the chain rule gives:

$$Tf \circ (\frac{d}{dt}\Big|_{t=0} \Phi_{exp(tX)}) = (\frac{d}{dt}\Big|_{t=0} \Psi_{exp(tX)}) \circ f$$

that is

$$Tf \circ X_M = X_N \circ f.$$

□

Proposition 4. The momentum map $\mu : M \to \mathfrak{g}^*$ is an Poisson map, which in this case means that

$$\{f \circ \mu, g \circ \mu\}_M(x) = \{f, g\}_{\mathfrak{g}^*}(\mu(x))$$

where $x \in M,\ f, g : \mathfrak{g}^* \to \mathbb{R}$ are two maps.

Proof. By density of polynomials in C^∞ functions, it is enough to prove this for two polynomials f and g. Using the Leibniz rule, we can even assume that f and g are linear functions on $\mathfrak{g}^*$, that is $f = X \in \mathfrak{g}, g = Y \in \mathfrak{g}$.

For such an f, the Hamiltonian vector field of the composed map $f \circ \mu$ is defined by

$$\begin{aligned} \omega_x(Z, X_{f\circ\mu}) &= (df)_{\mu(x)} T_x\mu(Z) \\ &=< T_x\mu(Z), X >= d\tilde{\mu}_X(x)(Z) \\ &= \omega_x(Z, \underline{X}), \end{aligned}$$

so that $X_{f\circ\mu} = \underline{X}$ and $X_{g\circ\mu} = \underline{Y}$, we have now

$$\begin{aligned} \{f \circ \mu, g \circ \mu\}(x) &= \omega_x(\underline{X}_x, \underline{Y}_x) \\ &=< T_x\mu(\underline{X}_x), Y > \\ &=< \underline{X}_{\mu(x)}, Y > \quad \text{(G-equivariant)}, \end{aligned}$$

as above. On the other hand,

$$\{f, g\}(\mu(x)) =< \mu(x), [X, Y] >= - < X_{\mathfrak{g}^*}, Y >=< \underline{X}_{\mu(x)}, Y > .$$

□

3.2 Existence and Uniqueness of Moment Map

Uniqueness of Moment Map

Let G be a compact Lie group.

Theorem 1. If $H^1(\mathfrak{g}; \mathbb{R}) = 0$, then moment maps for any Hamiltonian G-action is unique.

Proof. We suppose that $\tilde{\mu}^1$ and $\tilde{\mu}^2$ are two comoment maps for a Hamiltonian action. By definition for each $X \in \mathfrak{g}$, $\tilde{\mu}^1_X$ and $\tilde{\mu}^2_X$ are both hamiltonian functions for $\underline{X}$, thus $\tilde{\mu}^1_X - \tilde{\mu}^2_X = c(X)$ is a locally constant, and thus a constant on M (we will always assume that M is connected). So we get an element $c \in \mathfrak{g}^*$ by

$$< c, X >= c(X) = c_X$$

Since $\tilde{\mu}^1, \tilde{\mu}^2$ are Lie algebra homomorphism, we have

$$\begin{aligned} c_{[X,Y]} &= \tilde{\mu}^1_{[X,Y]} - \tilde{\mu}^2_{[X,Y]} = \{\tilde{\mu}^1_X, \tilde{\mu}^1_Y\} - \{\tilde{\mu}^2_X, \tilde{\mu}^2_Y\} \\ &= \{\tilde{\mu}^2_X + c_X, \tilde{\mu}^2_Y + c_Y\} - \{\tilde{\mu}^2_X, \tilde{\mu}^2_Y\} \\ &= 0 \end{aligned}$$

Note that in this case the two moment maps are related by

$$\mu^1 - \mu^2 = c,$$

in other words, they differed by a constant in $\mathfrak{g}^*$. Thus for $\forall X, Y \in \mathfrak{g}, c_{[X,Y]} = 0$, i.e. $c \in [\mathfrak{g}, \mathfrak{g}]^0 = H^1(\mathfrak{g}; \mathbb{R}) = 0$. So we get $\mu^1 = \mu^2$. □

Remark 4. In other words, moment map are unique up to elements of dual of the Lie algebra which annihilate the commutator ideal.

The two extreme cases are:

- G is semi-simple: any symplectic action is hamiltonian, moment maps are unique.
- G is commutative: symplectic action may not be hamiltonian, moment maps are unique up to any constant $c \in \mathfrak{g}$.

Existense of Moment Maps

We want to know when is that a Symplectic action is Hamiltonnian action? There are two answers, from the aspect of the smooth manifold M and Lie group G respectively.

Theorem 2. Suppose (M, ω) is a connected compact symplectic manifold with $H^1(M; \mathbb{R}) = 0$, then any symplectic action is hamiltonian.

Proof. Since $H^1(M; \mathbb{R}) = 0$, any symplectic vector field is hamiltonian vector field. So we can choose a basis $\{X_1, \cdots, X_k\}$ of $\mathfrak{g}$, for each X_i , we can find a function $\tilde{\mu}_{X_i}$ on M with

$$i_{\underline{X_i}}\omega = -d\tilde{\mu}_{X_i}.$$

For any $X \in \mathfrak{g}$, one can write

$$X = \sum_{i=1}^{k} = \lambda_i X_i$$

and we define

$$\tilde{\mu}_X = \sum_{i=1}^{k} = \lambda_i \tilde{\mu}_{X_i}.$$

This define a linear map $\tilde{\mu} : \mathfrak{g} \to C^\infty(M)$ with

$$i_{\underline{X}}\omega = -d\tilde{\mu}_X \quad \text{(i and d are both linear)}$$

In other words, the G-action is a weakly hamiltonian action, we should prove that $\tilde{\mu}$ is a Lie algebra morphism.

We consider the function,

$$\begin{aligned} c_{[X,Y]} &= \tilde{\mu}_{[X,Y]} - \{\tilde{\mu}_X, \tilde{\mu}_Y\} \\ dc_{[X,Y]} &= d\tilde{\mu}_{[X,Y]} - d\{\tilde{\mu}_X, \tilde{\mu}_Y\} \\ &= -i_{[X,Y]}\omega + i_{[X,Y]}\omega \\ &= 0 \end{aligned}$$

thus the function $c_{[X,Y]}$ is actually a constant, we can reduce it to 0 by some hypothesis. □

Theorem 3. Let G be a connected Lie group with

$$H^1(\mathfrak{g}; \mathbb{R}) = H^2(\mathfrak{g}; \mathbb{R}) = 0,$$

then any symplectic G-action is Hamiltonian.

Proof. First note that $H^1(\mathfrak{g}; \mathbb{R}) = 0 \Leftrightarrow [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.So any fundamental vector field $\underline{X}$ can be written as a summation of vector field of the form $[\underline{Y}, \underline{Z}]$, which is hamiltonian since the Lie bracket of any two symplectic vector fields is Hamiltonian.

We take an arbitary vector space lift $\tau : \mathfrak{g} \to C^\infty(M)$ for each basis vector $X \in \mathfrak{g}$, we choose $\tau(X) = \tau_X \in C^\infty(M)$ such that

$$X_{\tau_X} = \underline{X}.$$

The map $X \mapsto \tau_X$ may not be a Lie algebra morphism. By construction, $\tau_{[X,Y]}$ is hamiltonian function for $\underline{[X,Y]}$, and $\{\tau_X, \tau_Y\}$ is a hamiltonian function for $[\underline{X}, \underline{Y}]$. Since $[\underline{X}, \underline{Y}] = \underline{[X,Y]}$,

$$\tau_{[X.Y]} - \{\tau_X, \tau_Y\} = c(X,Y) \in \mathbb{R}.$$

By the jacobi identity, $\delta c = 0$.Since $H^2(\mathfrak{g}; \mathbb{R}) = 0$, there is a $b \in \mathfrak{g}^*$ satisfying $c = \delta b$, we define

$$\begin{aligned} \tilde{\mu} : \mathfrak{g} &\to C^\infty(M) \\ X &\mapsto \tilde{\mu}_X := \tau_X + b(X) \end{aligned}$$

Now, we obtained that

$$\tilde{\mu}_{[X,Y]} = \tau_{[X,Y]} + b([X,Y]) = \{\tau_X, \tau_Y\} = \{\mu_X, \mu_Y\}$$

So $\tilde{\mu}$ is a Lie algebra homomorphism. □

Corollary 1. If G is semi-simple, then any symplectic action is hamiltionian.

3.3 Classical Examples

Example 1. If $H : M \to \mathbb{R}$ is any function and if the Hamiltonian vector field X_H is complete, its flow defines a Hamiltonian $\mathbb{R}-$action, the momentum map of which is H.

Example 2. The circle S^1 acts on $\mathbb{C}^n$ by $u \cdot (z_1, \cdots, z_n) = (uz_1, \cdots, uz_n)$, that is

$$\begin{aligned} \Phi : S^1 \times \mathbb{C}^n \to \mathbb{C}^n \\ (e^{i\theta}, (z_1, \cdots, z_n)) \mapsto (e^{i\theta} \cdot z_1, \cdots, e^{i\theta} \cdot z_n) \end{aligned}$$

The fundamental vector field associated with $\frac{\partial}{\partial\theta} \in T_1S^1$ is

$$\begin{aligned} \underline{X}(z) &= \left.\frac{d}{dt}\right|_{t=0} (e^{it} \cdot z_1, \cdots, e^{it} \cdot z_n) \\ &= (i \cdot z_1, \cdots, i \cdot z_n) \end{aligned}$$

Since the standard symplectic form $\omega = dx \wedge dy$ on $\mathbb{C}^n$, we can get

$$\underline{X} = \sum_{j=1}^{n} (-y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j}).$$

So we compute $\tilde{\mu}_{\frac{\partial}{\partial\theta}}$,

$$\begin{aligned} i_{\underline{X}}\omega &= -d\tilde{\mu}_{\frac{\partial}{\partial\theta}} \\ \sum_{i=1}^{n} dx_i \wedge dy_i (\sum_{j=1}^{n} (-y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j}), Y) &= \sum_{j=1}^{n} (-y_j dy_j - x_j dx_j)(Y) \\ \implies d\tilde{\mu}_{\frac{\partial}{\partial\theta}} &= \sum_{j=1}^{n} (x_j dx_j + y_j dy_j). \end{aligned}$$

The moment map for this action is the hamiltonian

$$\mu = \frac{1}{2} \sum_{j=1}^{n} |z_j|^2.$$

Example 3. (Linear momentum). Consider a translation in $\mathbb{R}^3$, the phase space is $T^*\mathbb{R}^3 \cong \mathbb{R}^6 = (q_1, q_2, q_3, p_1, p_2, p_3)$ with the symplectic form $\omega = \sum_{i=1}^{3} dq_i \wedge dp_i$. Then the action is

$$\begin{aligned} &\Phi : \mathbb{R}^3 \times \mathbb{R}^6 \longrightarrow \mathbb{R}^6 \\ &\quad (\vec{a}, (\vec{q}, \vec{p})) \mapsto (\vec{q} - \vec{a}, \vec{p}), \qquad \text{where} \quad \vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3. \end{aligned}$$

So the fundamental vector field associated $X = \vec{a}$ is,

$$\underline{X} = \left.\frac{d}{dt}\right|_{t=0} (\vec{q} - \exp(t\vec{a}), \vec{p}) = \sum_{i=1}^{3} (-a_i \frac{\partial}{\partial q_i}).$$

Now we compute the function $\tilde{\mu}_X$ for it, that is $-d\tilde{\mu}_{\vec{a}} = i_{\underline{X}}\omega$.

$$\begin{aligned}
i_{\underline{X}}\omega(Y) &= \sum_{i=1}^{3} dq_i \wedge dp_i(\underline{X}, Y) \\
&= -\vec{a} \cdot d\vec{q}(Y) \\
&= -d(\vec{a} \cdot \vec{q})(Y)
\end{aligned}$$

So $\tilde{\mu}_{\vec{a}}(\vec{q}, \vec{p}) = \vec{a} \cdot \vec{q}$, then we get the moment map from

$$\tilde{\mu}_{\vec{a}}(\vec{q}, \vec{p}) =< \mu(\vec{q}, \vec{p}), \vec{a}) >= \vec{a} \cdot \vec{p}$$

$\implies \mu(\vec{q}, \vec{p}) = \vec{p}$, which is exactly the linear momentum.

Example 4. (Angular momentunm) Let $G = SO(3)$ act on $\mathbb{R}^3$ by

$$\Phi(A, \vec{q}) = A \cdot \vec{q}.$$

we consider the cotangent lift of this action which is symplectic action on cotangent bundle $\mathbb{R}^6$. The infinitesimal version of this action is

$$\begin{aligned}
\hat{\Phi} : \mathrm{so}(3) \times \mathbb{R}^6 &\to \mathbb{R}^6 \\
(X, (\vec{q}, \vec{p})) &= (X \cdot \vec{q}, X \cdot \vec{p})
\end{aligned}$$

Since the map $f : (a_1, a_2, a_3) \mapsto \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$ gives an isomorphism from so(3) to $\mathbb{R}^3$, we can compute the map

$$\begin{aligned}
\varphi : \mathbb{R}^3 \times \mathbb{R}^6 &\to \mathbb{R}^6 \\
(\vec{a}, (\vec{q}, \vec{p})) &\mapsto (\vec{a} \times \vec{q}, \vec{a} \times \vec{p})
\end{aligned}$$

where $\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$.

$$\begin{aligned}
\underline{\vec{a}} &= \frac{d}{dt}\bigg|_{t=0} \varphi(\exp(t\vec{a}), (\vec{q}, \vec{p})) \\
&= (\vec{a} \times \vec{q}, \vec{a} \times \vec{p})(\frac{\partial}{\partial q_1}, \cdots, \frac{\partial}{\partial p_3})^{\mathsf{T}}
\end{aligned}$$

Then

$$\begin{aligned}
-i_{\underline{\vec{a}}}(Y) &= \sum_{i=1}^{3} (dq_i \wedge dp_i)(\underline{\vec{a}}, Y) \\
&= (\vec{a} \times \vec{q} \cdot d\vec{p} - \vec{a} \times \vec{p} \cdot d\vec{q})(Y) \\
&= \vec{a} \cdot (\vec{q} \times d\vec{p} + d\vec{q} \times \vec{p})(Y) \\
&= \vec{a} \cdot d(\vec{q} \times \vec{p})(Y)
\end{aligned}$$

so that

$$\begin{aligned}
\tilde{\mu}_{\vec{a}}(\vec{q}, \vec{p}) &=< \mu(\vec{q}, \vec{p}), \vec{a} > \\
\vec{a} \cdot (\vec{q} \times \vec{p}) &= \mu(\vec{q}, \vec{p}) \cdot \vec{a}
\end{aligned}$$

$\implies \mu(\vec{q}, \vec{p}) = \vec{q} \times \vec{p}$, which is exactly angular momentum.

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