Symmetry in Symplectic Geometry

Shanzhong SUN

Department of Mathematics
Capital Normal University

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Outline

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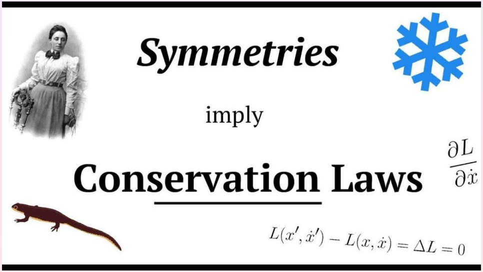
Symmetry in Symplectic Geometry

- in symplectic geometry, we study motions in phase space of classical mechanical systems (i.e., Hamiltonian systems on symplectic manifolds)
- Almost all mechanical systems have symmetries which imply constants of motions (=conservative quantities, i.e., first integrals of HS) according to E. Noether
- Slogan (Yang): Symmetry determines interaction!

Emmy Noether: Creative Mathematical Genius (1882-1935)



Emmy Noether: Creative Mathematical Genius (1882-1935)

- E. Noether, Invariante Variationsprobleme, Nachr. König. Gesell. Wissen. Göttingen, Math.-Phys. Kl. (1918), 235-257
- Y. Kosmann-Schwarzbach, The Noether Theorems. Invariance and Conservation Laws in the Twentieth Century, Springer, New York, 2011 (Including the English translation of Noether's classical paper above)

Classical Examples

Following Noether,

- space translation invariance $\Rightarrow$ conservation of linear momentum
- time translation invariance $\Rightarrow$ conservation of energy
- rotational invariance $\Rightarrow$ conservation of angular momentum

Classical Examples

- Harmonic Oscillator
- pendulum (simple(Audin p88, p90, p93), spherical(Audin, p103) and magnetic spherical)
- rigid bodies and tops (motions on Lie groups)
- N-body problem
- Fixed center problem
- geodesics
- ....

Classical Examples

Ref: Cushman-Bates, Global Aspects of Classical Integrable Systems

Some are even integrable systems!!!

- Harmonic Oscillators
- pendulums (simple, spherical (da Silva(p178)) and magnetic spherical)
- geodesics on S^3
- Kepler problem
- tops (Euler, Lagrange, Kovalevskaya)
- two-center problem a la Euler
- ....

How good are they?

How to prove nonintegrability? 3BP

Symmetry in Symplectic Geometry

Other reasons other than the symmetry in mechanics

- ”orbit methods” (Kostant, Kirillov,...) use SG in an essential way to construct representations. Fundamental objects are the coadjoint orbits which are naturally symplectic
- Hamiltonian action $\Rightarrow$ functions on the symplectic manifolds $\Rightarrow$ play the game of Morse theory in SG

Symmetry in Symplectic Geometry

- Geometric and Topological Aspects
- Dynamical Aspects (stability)

References

- J. E. Marsden and T. Ratiu, Introduction to Mechanics and Symmetry. A basic exposition of Classical Mechanical Systems. Texts in Applied Mathematics **17**, Springer-Verlag, New York, 1994
- J.-M. Souriau, Structure des systèmes dynamiques, Dunod, Paris, 1969
- V. I. Arnold, Mathematical methods in classical mechanics, GTM 60

Group Action

- Group: compact Lie groups/algebraic groups/finite groups/discrete groups, $\mathbf{R}$, S^1, $SO(n)$, $SU(n)$, $Sp(2n)$ and their complexification; discrete group
- Space M: manifold with various additional structures
- transformation group of the space: $Aut(M)$ (e.g. $Homeo(M)$, $Diff(M)$, $Sym(M, \omega)$, $Ham(M, \omega)$)
- G group action on M=group homomorphism $G \to Aut(M)$ compatible with additional structures
- Dynamical Systems: $G = \mathbf{R}$
- G group action on G itself: left, right, adjoint, coadjoint actions

Group Action

A way to construct new spaces

- orbit, orbit map and fundamental vector fields
- isotropy group/ stabilizer
- transitive action
- free action, locally free action (isotropy group discrete)
- effective action: each group element $g \neq e$ moves at least one $p \in M$
- quotient space=orbit space: topological (Hausdorff?) or additional properties?
- slice theorem (equivariant tubular neighborhood theorem)
- G group action on G: homogeneous space, flag variety

Group Actions: Examples

Ref: Audin, p11-12, p18 and exercise(p37)

Outline

Equivariant Cohomology: universal principal G-bundle

Ref: Audin (p178)

- universal: a numerable principal G-bundle $\mathcal{E} \to \mathcal{B}$ is called universal if (1) for any numerable principal G-bundle $E \to B$, there exists a map $f : B \to \mathcal{B}$ such that E is isomorphic to $f^*\mathcal{E}$; (2) two maps $f, g : B \to \mathcal{B}$ induce isomorphic bundles iff they are homotopic
- Milnor Join: Milnor's beautiful and explicit construction of universal principal G-bundle $EG \to BG$ (Audin p179)
- EG is contractible and the action of G on EG is free; BG the classifying space
- $BU(k) = G_k(\mathbf{C}^\infty)$ the infinite Grassmannian

Equivariant Cohomology: Borel Construction

- G acts on $M \times EG$ by $g \cdot (x, e) = (g \cdot x, g \cdot e)$ freely (since G acts on the second factor freely)
- Borel construction: $M_G := M \times_G EG$
- M_G is the homotopy theoretical quotient of M which is good enough and has the same homotopy type as M/G when the genuine quotient M/G is "good" (M/G is smooth)
- G-action free, $M_G = M/G$; G-action trivial, $M_G = BG \times M$, in particular, we have $M_G = BG$ when M is a point!

Equivariant Cohomology

- $H_G^*(M) := H^*(M_G)$
- $H_G^*(M)$ is a ring, further, it is an $H_G^*(pt)$-module
- For $G = S^1$, $BS^1 = \mathbf{CP}^\infty$ and $H^*(BS^l)$ is a polynomial ring on a generator u of degree 2 (Audin, p186)

Equivariant Cohomology: various models

Ref: Guillemin-Sternberg: SUSY and equivariant de Rham for differentiable manifolds M with Lie group G acting on it, equivariant version of de Rham theory

- naïve guess: $\Omega(M) \otimes \Omega(EG)$ corresponding to $M \times EG$ then take G-invariant part, hard to treat the 2nd factor due to ∞-dim.
- idea(1): introduce Lie supergroup G^* whose underlying manifold is G and underlying algebra is

$$\tilde{\mathfrak{g}} := \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

with $\mathfrak{g}_{-1}$ generated by ι_{X_a}, $\mathfrak{g}_0$ generated by L_{X_a} and $\mathfrak{g}_1$ by de Rham differential d. X_a fundamental vector field corresponding to the basis of $\mathfrak{g}$

Equivariant Cohomology: various models

for differentiable manifolds M with Lie group G acting on it, equivariant version of de Rham theory

- idea(2): replace $\Omega(EG)$ by an commutative graded superalgebra A equipped with a representationof G^* and such that
  - acyclic w.r.t. d (corresponding to *EG* is contractible)
  - $\exists\ \theta^b \in A^1$ s.t. $\iota_{X_a}\theta^b = \delta^b_a$ (corresponding to *G* acting on *EG* locally freely)
- we substitute for $\Omega(M) \otimes \Omega(EG)$ the algebra $\Omega(M) \otimes A$
- then the complex $(\Omega(M) \otimes A)_{basic}$ replaces $\Omega(M_G)$; basic=G-invariant+annihilated by ι_{X_a}

Equivariant Cohomology: various models

for differentiable manifolds M with Lie group G acting on it, equivariant version of de Rham theory

to be checked

- independent of A
- gives the right answer: the cohomology of the complex $(\Omega(M) \otimes A)_{basic}$ is isomorphic to $H_G^*(M)$

Equivariant Cohomology: various models

for differentiable manifolds M with Lie group G acting on it, equivariant version of de Rham theory

- Weil model: introduce Weil algebra $W = \wedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*)$; the model is $H^*((\Omega(M) \otimes W)_{basic}, d)$
- Cartan model: $H^*((\Omega(M) \otimes S(\mathfrak{g}^*))^G, d_G)$
- They are the SAME!

Equivariant Cohomology: SUSY perspective

- Mathai-Quillen construction of a universal equivariant Thom form
- Fermionic/super Fourier transform, Berezin integral
- V. Mathai and D. Quillen, Superconnections, Thom classes and equivariant differential forms, Topology **25**(1986), no. 1, 85-110
- J. Kalkman, A BRST model applied to symplectic geometry, Thesis, Utrecht (1993)

Localization Theorem: Torus action

Ref: Audin (p204)
For torus T-action, Forgetting torsion, the H^*BT-module $H^*_T(M)$ looks very much like the free H^*BT-module $H^*_T(F)$ with F fixed points

Theorem

Let $i : F \hookrightarrow M$ be the inclusion of fixed points of the action of a torus T on a manifold M. Then the supports of both the kernel and cokernel of

$$i^* : H^*_T(M) \to H^*_T(F)$$

are included in $\bigcup_{H \text{ stabilizer} \neq T} \mathfrak{h}$

Localization formula: Torus action

Ref: Audin (p207)

- If $x \in H_T^*(M)$, in a suitable localization,

$$x = \sum_{Z \subset F} \frac{i_*^Z i_Z^* x}{e_T(\nu_Z)}$$

S^1-euivariant Cohomology

Ref: da Silva (p229)

References

- M. Atiyah and R. Bott, The moment map and equivariant cohomology, Topology **23** (1984), 1-28
- M. Audin, Torus action on symplectic manifolds, PM **93**, Birkhäser Verlag, Basel, 2004
- J. J. Duistermaat, Euivariant cohomology and stationary phase, *Symplectic Geometry and Quantization* (Sanda and Yokohama, 1993), edited by Y. Maeda, H. Omori and A. Weinstein, 45-62, Contemp. Math. **179**, AMS Providence, 1994

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- V. Guillemin, Moment maps and Combinatorial Invariants of Hamiltonian T^n-spaces, PM **122**, Birkhäser, Boston, 1994
- V. Guillemin and S. Sternberg, Supersymmetry and Equivariant de Rham Theory, with an appendix containing two reprints by H. Cartan, Mathematics Past and Present, Springer-Verlag, Berlin, 1999

Outline

Canonical Symplectic Form

Ref: da Silva (p163); Audin (p60); Marsden-Ratiu

- Lie-Poisson or Kostant-Kirillov-Souriau: for $f, g \in C^\infty(\mathfrak{g}^*)$ and $\xi \in \mathfrak{g}^*$
$$\{f, g\}(\xi) := \langle \xi, [df_\xi, dg_\xi] \rangle$$
where $df_\xi : T_\xi \mathfrak{g}^* = \mathfrak{g}^* \to \mathbf{R}$ is identified to an element of $\mathfrak{g}$
- the symplectic form

$$\omega_\xi(X, Y) = \langle \xi, [X, Y] \rangle$$

Hermitian matrices

Ref: da Silva (p156), Audin

- unitary group $U(n)$ acts on the space of $n \times n$ complex Hermitian matrices

Outline

Hamiltonian Group Actions

- symplectic actions
- Hamiltonian actions
- symplectic v.s. Hamiltonian

Moment Map

- $\mu : M \to \mathfrak{g}^*$

Canonical Moment Map

- Lie group G acts on the symplectic manifold $G \cdot \xi$ for $\xi \in \mathfrak{g}^*$
- the inclusion $G \cdot \xi \subset \mathfrak{g}^*$ is a moment map for the G-action on its coadjoint orbit $G \cdot \xi$
- this is the factory/motherland for examples of moment map

Noether Theorem in the light of moment map

Ref: Audin, p79

Theorem

Let H be a function on M which is invariant under the G-action. Thus μ is constant on the trajectories of the Hamiltonian vector field X_H.

write all the classical conservation laws in terms of moment map!!!

Examples of Momemt Map

Ref: da Silva (p191)

- coadjoint orbit of unitary group (Audin, p 76)
- Complex Grassmannian (Audin, p101)
- symplectic teardrop (symplectic orbifold, Audin, p101)
- weighted projective spaces (Audin p102)
- symplectic cutting due to Lerman: a simple and elegant construction of new symplectic manifold out of Hamiltonian S^1 action and symplectic reduction (Audin p102, p139)

Existence of Moment Maps

Ref: da Silva (p195)

Theorem

If $H^1(\mathfrak{g}, \mathbf{R}) = H^2(\mathfrak{g}, \mathbf{R}) = 0$, then any symplectic G-action is Hamiltonian

- A compact Lie group G is semisimple if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$
- (Whitehead Lemma) Let G be a compact Lie group, then G is semisimple iff $H^1(\mathfrak{g}, \mathbf{R}) = H^2(\mathfrak{g}, \mathbf{R}) = 0$
- So if G is semisimple, then any symplectic G-action is Hamiltonian

Uniqueness of Moment Maps

Ref: da Silva(p196)

Theorem

For a compact Lie group G, if $H^1(\mathfrak{g}, \mathbf{R}) = 0$*, then moment maps for Hamiltonian G-actions are unique*

- In general, moment maps are unique up to elements of the dual of the Lie algebra which annihilate the commutator ideal
- one extreme: G semisimple: any symplectic action is Hamiltonian and moment maps are unique
- another extreme: G commutative: symplectic actions may not be Hamiltonian, moment maps are unique up to any constant $c \in \mathfrak{g}^*$

Outline

Marsden-Weinstein-Meyer Theorem

Ref: da Silva

Theorem (Marsden-Weinstein, Meyer)

Let (M, ω, G, μ) be a Hamiltonian G-space for a compact Lie group G. Let $i : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. Assume that G acts freely on $\mu^{-1}(0)$. Then

- *the orbit space $M_{red} = \mu^{-1}(0)/G$ is a smooth manifold*
- *$\pi : \mu^{-1}(0) \to M_{red}$ is a principal G-bundle*
- *there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega = \pi^*\omega_{red}$*

$(M_{\mathrm{red}}, \omega_{\mathrm{red}})$: reduced space, symplectic quotient, Marsden-Weinstein-Meyer quotient, symplectic reduction

Variants

- Lagrangian Reduction
- tangent and cotangent bundle reduction
- semidirect product reduction $SE(3) = SO(3) \ltimes \mathbf{R}^3$
- Routh reduction
- reduction by stages and group extensions
- singular reduction
- multisymplectic reduction
- discrete mechanical system motivated by numerical analysis
- ....
- Kähler (Atiyah) and hyperKähler (Hitchin) reduction

Reduction of Dynamics

Ref: da Silva (p173)

Examples of Reduction

Ref: da Silva (p198)

- symplectic cuttings

Very Brief History on Symplectic Reduction

Ref: Marsden-Weinstein, Some comments on the history, theory and applications of symplectic reduction, 2001

For moment map

- S. Lie: (1) an action of a Lie group G with Lie algebra $\mathfrak{g}$ on a symplectic manifold M should be accompained by a map $\mu : M \to \mathfrak{g}^*$ equivariant w.r.t. the coadjoint action (2)the orbits are symplectic
- In old days, links with mechaniccal system with symmetries: Euler, Lagrange, Hamilton, Poisson, Jacobi, Routh, Riemann, Liouville, Lie, Poincaré, Noether.
- moment map and its equivariance in modern form: Kostant (1966), Souriau (1966, 1970) in the general symplectic

Very Brief History on Symplectic Reduction

For moment map

- S. Smale, Topology and mechanics, Invent. Math. **10**(1970), 305-331; **11**(1970), 45-64 (apply topology, especially Morse theory, to study relative equilibria; lifted action from a manifold N to its cotangent bundle T^*N)
- about terminology: Smale (angular momentum); Souriau (application moment); Marsden-Weinstein (moment); Duistermaat-Cushman (momentum), Marsden-Weinstein/Abraham-Marsden (since 1976, momentum map/mapping); Guillemin-Sternberg (moment map/mapping)
- more history on moment map: Marsden-Ratiu (Introduction to mechanics and symmetry, 1999)

Very Brief History on Symplectic Reduction

Ref: Marsden-Weinstein, Some comments on the history, theory and applications of symplectic reduction, 2001

For symplectic reduction

- For G abelian, there are many precursors: Lagrange, Poisson, Jacobi, Routh
- Smale's observation: Jacobi's "elimination of the node" in $SO(3)$ symmetric problems is best understood as the division of a nonzero angular momentum level by the $SO(2)$ subgroup that fixes the momentum value
- For cotangent bundle: Smale clearly stated that coadjoint isotropy group G_k of $k \in \mathfrak{g}^*$ leaves $\mu^{-1}(k)$ invariant, but he only divided by G_k after fixing the total energy as well, in order to obtain the "minimal" manifold on which to analyze the reduced dynamics

Very Brief History on Symplectic Reduction

For symplectic reduction

- Marsden-Weinstein (1974): combine Souriau's momentum map for general symplectic actions, Smale's idea of dividing the momentum level by the coadjoint isotropy group, and Cartan's idea of removing the degeneracy of a 2-form by passing to the leaf space of the form's null foliation
- Meyer (1973) and Marsden-Weinstein (1974): the key observation was that the leaves of the null foliation are precisely the (connected components of the) orbits of the coadjoint isotropy group
- Marsden-Weinstein (1974): construction of the coadjoint orbits in $\mathfrak{g}^*$ by reduction of the cotangent bundle T^*G with its canonical symplectic structure

Know past, know future

- For masters like Euler, Lagrange..., their aim was to eliminate variables associated with symmetries in order to simplify calculations in concrete examples and much of these works were done with coordinate (Ref: Whittaker, 1907)

Know past, know future

- Routh (1860, 1884): reduction of systems with cyclic variables; Jacobi and Liouville (1870): reduction of systems with integrals in involution; modern Lagrangian reduction for the action of abelian groups
- Rigid body: Euler got the equiation aroud 1740; key example of coadjoint orbit reduction/ Euler-Poincaré reduction; Lagrange (1788) understood reduction as we do for $SO(3)$ today;

Know past, know future

- Jacobi's elimination of node for reducing the gravitational NBP by group $SE(3)$ of Euclidean motions around 1860
- it is related to work on rotating fluid masses held together by gravitation forces (star soup!) studied by Riemann, Jacobi, Poincaré,....
- Hidden in these examples is much of the beauty of modern reduction, stability and bifurcation theory for mechanical system with symmetries

Know past, know future

- both symplectic and Poisson geometry have their roots in the works of Lagrange, Jacobi and Poisson
- they matured at the hands of Lie, and he discovered many modern concepts, e.g., Lie-Poisson bracket on $\mathfrak{g}^*$
- How is it possible? Mystery!

Know past, know future

- notion of manifold: Lie, Poincaé, Weyl, Cartan, Reeb, Synge,.....
- it is time for a more general and intrinsic view of mechanics around 1960s

Know past, know future

- From 1960s, geometric mechanics exploded: Abraham, Arnold, Kirillov, Kostant, Mackey, MacLane, Segal, Sternberg, Smale, Souriau
- Kirillov and Kostant: deep connections between mechanics and pure math (orbit methods in group representations)
- Arnold and Smale were closer to mechanics
- Souriau: more in mathematical physics

Know past, know future

- modern vision of mechanics combines strong links to important questions in pure math with traditional classical mechanics of particles, rigid bodies, fields, fluids, plasmas......
- symmetries vary from obvious translational and rotational symmetries to less obvious particle relabeling symmetries in fluids, to the "hidden" symmetries underlying integrable systems

Know past, know future

- Arnold (1966) focused on systems whose configuration space is a Lie group (e.g. rigid body and fluid as did by Poincaré)
- Samle (1970) focused on bifurcations of relative equilibria (number and stability of relative equilibria in planar NBP by a topological study of the energy-momentum mapping)
- these paved the way for Meyer and Marsden-Weinstein

References

- K. Meyer, Symmetries and integras in mechanics, Dynamical Systems (Proc. Sympos. Univ. Bahia, Salvador, 1971), 259-272. Academic Press, New York, 1973
- J. E. Marsden and A. Weinstein, Reduction of Symplectic manifolds with symmetry, Rep. Mathematical Physics, **5** (1974), 121-130

Outline

Motivating Example

Ref: da Silva, Audin (p84)

- $G = \mathbf{C}\backslash\{0\}$ acts on $M = \mathbf{C}^n$:

$$\begin{aligned} \Psi : (\mathbf{C}\backslash\{0\}) \times \mathbf{C}^n &\rightarrow \mathbf{C}^n \\ (\lambda, z) &\mapsto \lambda \cdot z =: \Psi_\lambda(z) \end{aligned}$$

- orbits: through nonzero $z \in \mathbf{C}^n$, it is the punctured complex line $\mathbf{C}\backslash\{0\}$; through $0 \in \mathbf{C}^n$, it is just a single point which is "unstable"
- the orbit space is $M/G = \mathbb{CP}^{n-1} \sqcup \{point\}$
- the quotient topology restricts to the usual topology on $\mathbb{CP}^{n-1}$; the only open set containing $\{point\}$ in the quotient topology is the full space. The quotient topology in M/G is NOT Hausdorff!!

Motivating Example

- it suffices to remove 0 from $\mathbf{C}^n$ to obtain a Hausdorff orbit space: $\mathbb{CP}^{n-1}$ (GIT quotient)
- it is well known that $\mathbb{CP}^{n-1}$ has a COMPACT (but NOT COMPLEX) description

$$\mathbb{CP}^{n-1} = (\mathbf{C}^n \backslash \{0\})/(\mathbf{C} \backslash \{0\}) = S^{2n-1}/S^1$$

- the last step is in fact a symplectic quotient!

Motivating Example

- the standard symplectic structure on $\mathbf{C}^n$ is

$$\omega = \frac{i}{2}\sum dz_i \wedge d\bar{z}_i = \sum dx_i \wedge dy_i = \sum r_i dr_i \wedge d\theta_i$$

- consider the S^1 group action ψ on $(\mathbf{C}^n, \omega)$

$$\begin{aligned} \psi : S^1 \times \mathbf{C}^n &\rightarrow \mathbf{C}^n \\ (t = e^{i\theta}, z) &\mapsto t \cdot z \end{aligned}$$

- the action ψ is Hamiltonian action with moment map

$$\begin{aligned} \mu : \mathbf{C}^n &\rightarrow \mathbf{R} \\ z &\mapsto -\frac{|z|^2}{2} + cost. \end{aligned}$$

Motivating Example

- This is because

$$\iota_{X^\sharp}\omega = -\sum r_i dr_i = -\frac{1}{2}\sum dr^2 = d\mu$$

with $X^\sharp = \frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2} + \cdots \frac{\partial}{\partial\theta_n}$

- choose constant to be $\frac{1}{2}$, then $\mu^{-1}(0) = S^{2n-1}$, and the orbit space of the zero level set of the moment map is

$$\mu^{-1}(0)/S^1 = S^{2n-1}/S^1 = \mathbb{CP}^{n-1}$$

- we realize the GIT quotient $\mathbb{CP}^{n-1}$ as a symplectic quotient/reduced space!

Motivating Example

three major themes for Hamiltonian torus action

- (Marsden-Weinstein-Meyer) the reduced spaces are symplectic manifolds (whence the symplectic quotient)
- (Atiyah-Guillemin-Sternberg) the image of the moment map is a convex polytope
- (Delzant) Hamiltonian $\mathbb{T}^n$ spaces are classified by the image of the moment map

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- D. Mumford, J. Fogarty and F. Kirwan, Geometric Invariant Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete **34**, Springer-Verlag, Berlin, 1994
- F. Kirwan, Cohomology of Quotients in Symplectic and Algebraic Geometry, Mathematical Notes **31**, Princeton University Press, Princeton, 1984
- C. T. Woodward, Moment map and geometric invariant theory, arXiv: 0912.1132, 2009
- R. P. Thomas, Notes on GIT and symplectic reduction of bundles and varieties, arXiv: 0512411

Outline

Convexity theorem

Ref: da Silva (p199)

Theorem (Atiyah, Guillemin-Sternberg)

Let (M, ω) be a compact symplectic manifold, and $G = T^m = \mathbf{R}^m/\mathbf{Z}^m$ be an m-torus. Suppose that $\psi : T^m \to Symp(M, \omega)$ is a Hamiltonian action with moment map $\mu : M \to]R^m$. Then

- *The levels of μ are connected*
- *The image of μ is convex*
- *The image of μ is the convex hull of the images of the fixed points of the action*

The image $\mu(M)$ of the moment map is called moment polytope

Refinement

- Atiyah: Kähler version
- Hitchin et al: HyperKähler version
- noncommutative convexity theorem due to Kirwan

References

- M. Atiyah, Convexity and commuting Hamiltonians, Bull. London Math. Soc. **14** (1982) 1-15
- V. Guillemin and S. Sternberg, Convexity properties of the moment mapping, Invent. Math. **67** (1982), 491-513
- F. Kirwan, Convexity properties of the moment mapping III, Invent. Math. **77** (1984), 547-552

Examples

Ref: da Silva (p202), Audin (p137)

- torus action on (weighted) projective spaces
- Hirzebruch surfaces (Audin p137, p145)
- Blowing-up the projective space (Audin p138)
- torus action on the Hermitian matrices (Audin p141)
- Permutahedron (Audin, p142)

Applications

- Kushnirenko theorem about the enumeration of solutions to a particular system of algebraic equations (Audin, p4, p129)
- Toeplitz-Hausdorff theorem about the numerical image of an operator on a Hilbert space (Audin, p5, p113)
- Schur-Horn theorem about the possible values of the diagonal entries in a Hermitian matrix of a given spectrum (Audin, p6, p117)
- compact symplectic $SU(2)$-manifolds of dim. 4 due to Iglesias (Audin, p131)

Outline

Completely Integrable Hamiltonian System (CIHS)

- A CIHS is a collection of n independent (over an open dense subset) Poisson commuting functions defined over a symplectic manifold M^{2n}
- Mineur-Arnold-Liouville theorem: locally, near a compact connected component of a regular level of an CIHS, there is a Hamiltonian torus action
- action-angle coordinates (only defined locally in general. e.g., there are NO GLOBAL action-angle coordinates for spherical pendulum (Audin, p103))

CIHS v.s. Torus Action

- the moment map of an effective Hamiltonian torus action on a symplectic manifold of the right dimension is an CIHS
- On the other hand, there are many CIHS that do not come from integrable torus action, e.g. any nonconstant function on a surface is a CIHS

Pelayo-Vù Ngọc, semitoric IS, 2007-2011

- we focus on (M^4, ω) compact connected symplectic manifold
- the momentum map $F = (f_1, f_2) \in C^\infty(M, \mathbf{R}^2)$ gives a CIHS on M: (1) $\{f_1, f_2\} = 0$; (2) X_{f_1}, X_{f_2} linearly independent almost everywhere
- Def: CIHS $F = (f_1, f_2)$ is toric if the flows of X_{f_1} and X_{f_2} are 2π-periodic and the action

$$T^2 \times M \to M, ((t_1, t_2), x) \mapsto ((\phi_{f_1}^{t_1} \circ \phi_{f_2}^{t_2})(x))$$

  is effective
- According to Atiyah and Guillemin-Sternberg: $P = F(M) \subset \mathbf{R}^2$ is convex

Pelayo-Vũ Ngọc, semitoric IS, 2007-2011

Singularities of CIHS

- points where X_{f_1}, X_{f_2} are linearly dependent
- Eliasson: notion of nondegenerate singularity $\Rightarrow$ normal form
- $\exists$ local coordinates (x_1, x_2, ξ_1, ξ_2) such that $\omega = dx_1 \wedge d\xi_1 + dx_2 \wedge d\xi_2$ and $(f_1, f_2) \sim (q_1, q_2)$ where q_i are some of
  - $q_i = \xi$ (regular component)
  - $q_i = \frac{x_i^2 + \xi_i^2}{2}$ (elliptic component)
  - $q_i = x_i \xi_i$ (hyperbolic component)
  - $q_1 = x_1\xi_2 - x_2\xi_1, q_2 = x_1\xi_1 + x_2\xi_2$ (focus-focus component)

Pelayo-Vù Ngọc, semitoric IS, 2007-2011

focus-focus singularity

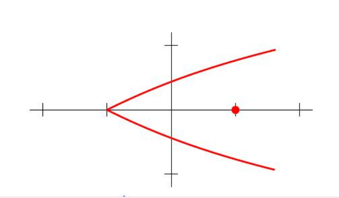
Pelayo-Vù Ngọc, semitoric IS, 2007-2011

- Def: Semitoric system: a CIHS $F = (J, H) : (M^4, \omega) \to \mathbf{R}^2$ is semitoric if
  - J is proper
  - the Hamiltonian flow of J yields an effective S^1-action
  - F has non-degenerate singularities only (like toric) with no hyperbolic component (these create problems, e.g., disconnected fibers)
- compared with toric case, focus-focus singularities appear
- for a semitoric system, $F(M)$ need NOT be a convex polygon

Pelayo-Vù Ngọc, semitoric IS, 2007-2011

An Example: Spin-Oscillator

- phase space $(M,\omega) = S^2_{(x,y,z)} \times \mathbf{R}^2_{(u,v)}, \omega = \omega_{S^2} \oplus \omega_{\mathbf{R}^2}$
- $F = (J,H)$: $J = \frac{u^2+v^2}{2} + z$ and $H = \frac{ux+vy}{2}$



Pelayo-Vù Ngọc, semitoric IS, 2007-2011

Another: Coupled angular momenta (Sadovki-Zhilinskií, 1999)

- $(M, \omega) = (S^2_{(x_1,y_1,z_1)} \times S^2_{(x_2,y_2,z_2)}, R_1\omega_{S^2} \oplus R_2\omega_{S^2})$, $R_1, R_2 > 0$, $X = x_1x_2 + y_1y_2$
- $J = R_1z_1 + R_2z_2$ and $H_t = (1 - t)z_1 + t(X + z_1z_2)$

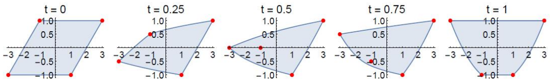


Figure: The momentum map image for varying values of t and $R_1 = 1$, $R_2 = 2$.

Pelayo-Vù Ngọc, semitoric IS, 2007-2011

- (M, ω, F) and (M', ω', F') semitoric are isomorphic $\Leftrightarrow$ $\exists \phi : M \to M'$ symplectomorphism and

$$g : F(M) \to F'(M'), g(x, y) = (x, g^{(2)}(x, y)), \frac{\partial g^{(2)}}{\partial y} > 0$$

such that $F' \circ \phi = g \circ F$

Pelayo-Vù Ngọc, semitoric IS, 2007-2011

Theorem (Pelayo-Vù Ngọc)

Semitoric systems are classified up to isomorphism through five invariants

- *the number m_f of focus-focus singula points*
- *a family of convex polygons obtained from $F(M)$*
- *the heights of the images of focus-focus points in these polygons*
- *a formal series for each focus-focus point*
- *an integer for each focus-focus point (twisting index)*

Outline

Toric Manifolds/Toric Varieties

Ref: Audin, VII §1

- Toric varieties were introduced by Demazure as closures of complex torus orbits in algebraic varieties
- For us, they are very beautiful families of symplectic manifolds endowed with Hamiltonian TORUS actions (even half-dimension). In fact, according to Delzant, they are the compact symplectic manifolds endowed with completely integrable torus actions!!!
- They play important roles in combinatorics (hence everything is explicit! polyhedron and fans), algebraic geometry, complex geometry, symplectic geometry, mathematical physics (string theory (Gromov-Witten on toric Calabi-Yau,...), gauge theory (gauged linear sigma model...))

Symplectic Reduction and Convex Polyhedron

- Construction of toric manifold X_Σ: convex polyhedron $P \Rightarrow$ Fan $\Sigma(P) \Rightarrow$ toric manifold X_Σ
- Conversely, we can associate a polyhedron to a toric manifold: Following Delzant, X_Σ can be endowed with symplectic forms for which the action of the compact torus is completely integrable and the image of the moment map is one of the convex polyhedra leading toe fan Σ——the shape of the polyhedron determines the fan and the volumes of its faces then determine the symplectic form
- Fact: All the primitive polyhedra are indeed images of moment map of completely integrable torus actions (This, together with the uniqueness theorem, constitute the classification theorem of Delzant, See Audin, Thm VII.2.1(p 245) with motivating example p244)

Compact Complex Toric Surfaces

Theorem (Oda)

Any compact complex toric surface is obtained from $\mathbf{CP}^2$ *or from a Hirzebruch surface by a finite sequence of blow ups*

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Symplectic Toric Manifold

- A $2n$-dim Symplectic toric manifold is a compact connected symplectic manifold (M^{2n}, ω) equipped with an effective Hamiltonian action of an n-torus T^n with the corresponding moment map $\mu : M \to \mathbf{R}^n$

Delzant Polytopes

Ref: da Silva, Audin

A Delzant polytope $\Delta \subset \mathbf{R}^n$ is a convex polytope such that

- it is simple: there are n edges meeting at each vertex
- it is rational: the edges meeting at the vertex p are rational in the sense that each edge is of the form $p + tu, t \geq 0, u \in \mathbf{Z}^n$
- it is smooth: for each vertex, the corresponding $u_1, ..., u_n$ can be chosen to be a $\mathbf{Z}$-basis of $\mathbf{Z}^n$

Note: it is closely related to Newton polytopes (which are the nonsingular n-valent polytopes), except that the vertices of a Newton polytope are required to lie on the integer lattice and for a Delzant polytope they are not

Delzant Theorem

classification theorem for symplectic toric manifold in terms of combinatorial data.

Theorem (Delzant)

Symplectic toric manifolds are classified by Delzant polytopes. More precisely, there is a one-to-one correspondence

$$\begin{array}{ccc} \{toric\ manifolds\} & \xrightarrow{1-1} & \{Delzant\ polytopes\} \\ (M^{2n}, \omega, T^n, \mu) & \mapsto & \mu(M) \end{array}$$

References about Delzant's Theorem

- T. Delzant, Hamiltoniens périodiques et images convexes de l'application moment, Bull. Soc. Math. France **116** (1988), 315-339
- Audin (p123)
- da Silva (p211)
- V. Guillemin, Moment maps and combinatorial invariants of Hamiltonian T^n-space, Progress in Mathematics, **112**, Birkhäuser, Boston, 1994

Outline

Duistermaat-Heckman Polynomial

Ref: da Silva (p223)

- (M, ω, G, μ) Hamiltonian G-space with G torus and μ proper
- Liouville/symplectic measure

$$m_\omega(\mathcal{U}) = \int_{\mathcal{U}} \frac{\omega^n}{n!}$$

for Borel subset $\mathcal{U}$ of M

- Duistermaat-Heckman measure, m_{DH} on $\mathfrak{g}^*$ is the pushforward of m_ω by $\mu : M \to \mathfrak{g}^*$, i.e.,

$$m_{DH}(U) := (\mu_* m_\omega)(U) = \int_{\mu^{-1}(U)} \frac{\omega^n}{n!}$$

for Borel subset $U \subset \mathfrak{g}^*$

Duistermaat-Heckman Polynomial

- for a compactly-supported function $h \in C_0^\infty(\mathfrak{g}^*)$, its integral w.r.t. m_{DH} is defined to be

$$\int_{\mathfrak{g}^*} h dm_{DH} := \int_M (h \circ \mu) \frac{\omega^n}{n!}$$

- regard $\mathfrak{g}^*$ as a vector space $\mathbf{R}^n$, there is also the Lebesgue measure m_0
- the relation between m_{DH} and m_0 is governed by the Radon-Nikodym derivative $\frac{dm_{DH}}{dm_0}$ which is a generalized function s.t.

$$\int_{\mathfrak{g}^*} h dm_{DH} = \int_{\mathfrak{g}^*} h \frac{dm_{DH}}{dm_0} dm_0$$

Duistermaat-Heckman Polynomial

Theorem (Duistermaat-Heckman,1982)

The Duistermaat-Heckman measure is a piecewise polynomial multiple of Lebesgue measure m_0 on $\mathfrak{g} \simeq \mathbf{R}^n$, that is the Radon-Nikodym derivative $f = \frac{dm_{DH}}{dm_0}$ is piecewise polynomial. More precisely, for any Borel subset U of $\mathfrak{g}^*$,

$$m_{DH}(U) = \int_U f(x)dx$$

where $dx = dm_0$ is the Lebesgue volume form on U and $f : \mathfrak{g}^* \simeq \mathbf{R}^n \to \mathbf{R}$ is polynomial on any region consisting of regular values of μ.

Duistermaat-Heckman Polynomial

- the Radon-Nikodym derivative f is called the Duistermaat-Heckman polynomial
- For toric, the Duistermaat-Heckman polynomial is a universal constant equal to $(2\pi)^n$ when the polytope Δ is n-dimensional
- So the symplectic volume $(M_\Delta, \omega_\Delta)$ (Delzant construction) is $(2\pi)^n$ times the Euclidean volume of Delzant polytope Δ

Variation of Symplectic Volume

A natural question about the symplectic reduction

- (M, ω, G, μ) Hamiltonian G-space with G an n-torus (could be general G, see Guillemin, PM 122) and μ proper: we take $G = S^1$ for simplicity
- Suppose that G acts freely on $\mu^{-1}(0)$, it also acts freely on nearby levels $\mu{-}1(t)$ for $t \in \mathfrak{g}^*, t \approx 0$
- consider the reduced spaces $M_{\text{red}} = \mu^{-1}(0)/G$ and $M_t = \mu^{-1}(t)/G$ with reduced symplectic forms ω_{red} and ω_t
- What is the relation between these reduced spaces as symplectic manifolds?

Variation of Symplectic Volume

Theorem (Duistermaat-Heckman,1982)

The cohomology class of the reduced symplectic form $[\omega_t]$ varies linearly in t. More specifically,

$$[\omega_t] = [\omega_{red}] = tc$$

where $c \in H^2_{de\ Rham}(M_{red})$ is the first Chern class of the S^1-bundle $Z \to M_{red}$

In general, once t_1 and t_2 lies in the same component of regular values of the moment map, the difference of cohomology classes $[\omega_{t_1}] - [\omega_{t_2}]$ is a linear function of $t_1 - t_2$
Example: Audin (p 192)

Variation of Symplectic Volume

- (M, ω, S^1, μ) Hamiltonian S^1-space of dim. $2n$ and (M_x, ω_x) be its reduced space at level x
- for $x \approx 0$, the symplectic volume of M_x

$$\mathrm{Vol}(M_x) = \int_{M_x} \frac{\omega_x^{n-1}}{(n-1)!} = \int_{M_{\mathrm{red}}} \frac{(\omega_{\mathrm{red}} - x\beta)^{n-1}}{(n-1)!}$$

  a polynomial in x of degree $n-1$
- in another way

$$\mathrm{Vol}(M_x) = \int_{Z} \frac{\pi^*(\omega_{\mathrm{red}} - x\beta)^{n-1}}{(n-1)!} \wedge \alpha$$

  α a chosen connection from the S^1-bundle $Z \to M_{\mathrm{red}}$, β its curvature form

Duistermaat-Heckman v.s. Equivariant Cohomology

- Berline-Vergne: the language of equivanriant cohomology fits very well with the study of Hamiltonian actions, since the existence of a moment map for the Hamiltonian G-action on the symplectic manifold M is equivalent to the existence of an extension of the symplectic form to the Borel construction M_G on M
- this is a perfect example of a theorem which becomes practically tautological once the right language to state it is found!

Duistermaat-Heckman v.s. Equivariant Cohomology

Ref: Audin, p189

Theorem

Let (M, ω) be a symplectic manifold endowed with a symplectic action of the Lie group G. Let $\mu : M \to \mathfrak{g}^*$ be any differentiable map. The formula

$$\omega^\sharp = \omega + d\langle\theta, \mu\rangle$$

defines a closed 2-form on the Borel construction M_G iff the G-action is Hamiltonian with moment map μ

Here $\theta \otimes \mu$ is a 1-form valued in $\mathfrak{g} \otimes \mathfrak{g}^*$, and may be contracted to give a 1-form valued in **R** which we denote $\langle\theta, \mu\rangle$

Duistermaat-Heckman v.s. Equivariant Cohomology

periodic Hamiltonian satisfies the ”Exact Stationary Phase Formula”

Theorem (Duistermaat-Heckman, 1982)

Let (M, ω) be a symplectic manifold of dim. $2n$ and let H be a periodic Hamiltonian on M with only isolated fixed points. Then

$$\int_M e^{-Hu} \frac{\omega^{\wedge n}}{n!} = \sum_{Z \in F} \frac{e^{-uH(Z)}}{e_{S^1}(\nu_Z)}$$

- isolatedness can be relaxed
- circle action can be replaced by torus action
- this is an equality of formal power series in the variable u and u^{-1}
- polynomial theorem is a corollary (Audin, p209)

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Moduli Spaces are the Spaces of the Future!

- Riemann, Mumford, Deligne, Griffths, Kodaira-Spencer
- moduli spaces appearing in gauge theory and string theory(moduli space of various connections, various bundles, pseudo-holomorphic curves): Witten, Donadson, Nahm, Hitchin, Ruan, Gromov-Witten, Floer, Fukaya.....(TQFT)
- derived shifted Poisson/Symplectic geometry

Differential Geometry required

- principal G-bundle
- connection and curvature
- holonomies
- characteristic classes

Atiyah-Bott, 1893, 1984

- The space $\mathcal{M} = \mathcal{M}_\Sigma(G) = \mathcal{M}_{g,d}(G)$ of all connections of a principal G-bundle over a compact oriented 2d Riemannian manifold with boundary or not (e.g. Riemann surface) may be treated as an infinite dimensional symplectic affine space
- gauge transformation group action is a Hamiltonian action with moment map the curvature!
- symplectic quotient is then the moduli space of flat (integrable) connections
- symplectic structure and Poisson structure: Audin p154

Integrable system on $\mathcal{M}$: Goldman, 1986

- Goldman functions on $\mathcal{M}$ and its Hamiltonian flow
- Poisson commutativity of Goldman functions (Goldman)
- counting of independent Goldman functions

Examples

- general surface of genus g with d holes and $G = S^1$ (Audin p152)
- $G = SU(2)$ (Audin Chapter V)
- three-holed sphere with $G = SU(2)$ (Audin p153, p167)/ one-holed torus (Audin p159, p167)

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Moduli space $\mathcal{M}$ of flat connections

- By holonomy construction, $\mathcal{M}$ can be identified with the space of homomorphisms from $\pi_1(\Sigma)$ to G modulo conjugation by elements of G (character variety)
- the latter space is also identified with a space of isomorphism classes of holomorphic vector bundles when Σ is a Riemann surface
- all of these spaces have symplectic structures!!

Hitchin Moduli Space

- Hitchin System of Equations (self-duality equations on a Riemann surface)

$$F_A + [\phi, \phi^*] = 0; \quad d''_A \phi = 0$$

- this is a dimensional reduction of the anti-self dual Yamg-Mills/instanton equations in dimension 4 (origion of HperKähler structure)
- this means that the $SU(n)$-connection A is compatible with the holomorphic structure of the bundle E
- F_A is the curvature of connection$A$
- $d''_A \phi$ is the anti-holomo. part of the covariant derivative of ϕ
- differential-geometric flavor system of nonlinear PDEs
- Hitchin equations are equivalent to the flatness of an $SL(n, \mathbf{C})$-connection $A + \phi + \phi^*$

WHY Hitchin Moduli Space?

- Rich structures
- play a role in many different areas including gauge theory, Kähler and HyperKähler geometry, surface group representations, integrable systems, nonabelian Hodge theory, mirror symmetry, geometric Langlands duality

Hitchin Moduli Space: Equivalent descriptions

- Hitchin moduli space $\mathcal{M}_H$=space of solutions to Hitchin system of nonlinear PDequations (=moment map for the action of the gauge group, HyperKähler reduction)
- moduli space $\mathcal{M}_{Dol}(\Sigma, n)$ of stable rank n degree 0 Higgs bundles (E, ϕ) on the Riemann surface Σ, here $\phi \in \Gamma(End(E) \otimes \Omega^1(\Sigma))$ is the Higgs field (Hitchin integrable systems)
- moduli space $\mathcal{M}_{DR}(\Sigma, n)$ of stable holomorphic connections on rank n holomorphic vector bundles $V \to \Sigma$ (isomonodromy systems)
- the moduli space $\mathcal{M}_B(\Sigma, n)$ (i.e., $\mathrm{Hom}(\pi_1(\Sigma), GL_n(\mathbf{C}))/GL_n(\mathbf{C})$ of irreducible $SL(n, \mathbf{C})$ complex representations of the fundamental group $\pi_1(\Sigma)$ (character variety, mapping class group actions)

Hitchin Moduli Space: Equivalent descriptions

- Hitchin-Kobayashi correspondence/principle (interpreting stability conditions for algebro-geometric objects as the condition for existence of solutions of gauge-theoretic PDEs) for Higgs bundles (Hitchin, Simpson):

$$\mathcal{M}_H \cong \mathcal{M}_{Dol}$$

- Nonabelian Hodge theory:

$$\mathcal{M}_{Dol}(\Sigma, n) \cong \mathcal{M}_{DR}(\Sigma, n)$$

- Riemann-Hilbert correspondence:

$$\mathcal{M}_{DR}(\Sigma, n) \cong \mathcal{M}_{B}(\Sigma, n)$$

Hitchin Moduli Space: beauties

- HyperKähler!
- integrable systems!

Hitchin Moduli Space:References

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Hitchin Moduli Space:References

To the world of meromorphic even irregular singular objects

- C. Simpson
- P. Boalch
- D. Gaiotto- G. Moore-A. Neitzke
- STOKES!!!

Term paper

- concrete examples of moment map and reduction
- integrable systems preferably with singularities!