Ex in Ch3.

Ex. T^n-action

Recall $\mathcal{U}(n)$-orbits in Hermitian metrices.

$\mathcal{U}(n)$ acts by conjugation on $\mathcal{H}$ of Hermitian metrices

$$\mathcal{H} = \{ X \in M_n(\mathbb{C}) \mid X^* = X \}$$

$\mathcal{H}_\lambda = \{ A \in \mathcal{H} \mid \text{spectrum of } A = (\lambda_1, \dots \lambda_n) \}$ for $\lambda \in \mathbb{R}^n$.

Consider $\lambda_1 \neq \lambda_2$, $\lambda = (\lambda_1, \lambda_2, \dots \lambda_2)$

$$\Psi_\lambda : \mathcal{H}_\lambda \longrightarrow \mathbb{C}P^{n-1} \quad \text{is diffeomorphism.}$$
$$A \longmapsto \lambda_1\text{-eigenline } l.$$

| prop III 1.10. $\mu : \mathcal{H}_\lambda \to t^* = \mathbb{R}^n$
$A \mapsto (a_{11}, a_{22}, \dots a_{nn})$
is the moment map for the T^n-action.

$$\underset{\sim}{\mathbb{C}P^{n-1}} \xrightarrow{\Psi_\lambda^{-1}} \mathcal{H}_\lambda \xrightarrow{\mu} t^* = \mathbb{R}^n.$$

$$[z_1, \dots z_n] \longrightarrow A$$

($(z_1, \dots z_n)$ is of unit length)

Let $e_1 = (z_1, \dots z_n)$, expand e_1 to unitary basis $\{e_1, e_2, \dots e_n\}$

$$A e_1 = \lambda_1 e_1, \quad A e_k = \lambda_2 e_k \quad (2 \le k \le n)$$

$$A(e_1, \dots e_n) = (e_1, \dots e_n) \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_2 \end{pmatrix}$$

$$A = (e_1, \dots e_n) \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_2 \end{pmatrix} \begin{pmatrix} e_1^* \\ \vdots \\ e_n^* \end{pmatrix}$$

diagnol of A is

$$(\lambda_1-\lambda_2)\,(|z_1|^2, |z_2|^2, \dots, |z_n|^2) + (\lambda_2, \dots \lambda_2)$$

Ex8. $\mathfrak{u}(n)$, $SU(n)$ action

(i) $U(n)\times\mathbb{C}^n\to\mathbb{C}^n$ by left multiplication

with comoment map $\tilde{\mu}_X(z) = \frac{i}{2} z^* X z$.

It defines a moment map $\mu:\mathbb{C}^n\to\mathfrak{u}(n)^*$

(ii) Deduce that the (induced) action of $U(n)$ on $\mathbb{C}P^{n-1}$ is Hamiltonian action and its moment map.

(iii) $SU(n)\subset U(n)$ $\quad \mathfrak{u}(n)^*\to \mathfrak{su}(n)^*$

$\mu:\mathbb{C}^n\to\mathfrak{su}(n)^*$, $\mu:\mathbb{C}P^{n-1}\to\mathfrak{su}(n)^*$.

(iv) $n=2$

$$S^2 \longrightarrow \mathbb{C}\cup\{\infty\}=\mathbb{C}P^1 \xrightarrow{\mu} \mathfrak{su}(2) \cong \mathbb{C}\times\mathbb{R}$$

$$(x,y,z) \longmapsto \frac{x+iy}{1-z} \longrightarrow \begin{pmatrix} a & b \\ \bar{b} & -a\end{pmatrix} \mapsto (b,a)$$

is inclusion $(x,y,z)\mapsto \frac{1}{2}(x,y,z)$.

pf. $\forall z\in\mathbb{C}^n$. $(\mathbb{R}^{2n})$. $z=(z_1,\dots,z_n)$ $\quad z_k = x_k+iy_k$

$\forall X\in\mathfrak{u}(n)$, $X = A+iB$. $\quad A,B\in M_n(\mathbb{R})$

$X+X^*=0 \Rightarrow A+A^T=0,\quad B=B^T.$

$\mathbb{C}^n$: $\omega_{std} = \Sigma dx_i\wedge dy_i$

Indentify $\mathbb{C}^n$, $T_z\mathbb{C}^n$, $T_z^*\mathbb{C}^n$ by standard inner product.

verify $\tilde{\mu}_X(z) = \frac{i}{2} z^* X z$ is comoment map :

$$d\tilde{\mu}_X(z) = (i_{\underline{X}}\,\omega_{std})_z.$$

$$\tilde{\mu}_X(z) = \frac{i}{2} z^* X z = \frac{i}{2}(x^T - iy^T)(A + iB)(x+iy)$$

$$= -\frac{1}{2}\langle x, Bx\rangle + \langle y, Ax\rangle - \frac{1}{2}\langle y, By\rangle$$

$$d\tilde{\mu}_X(z) = -\frac{1}{2}\langle dx, Bx\rangle - \frac{1}{2}\langle x, B\,dx\rangle + \cdots$$

$$\underline{X}_z = \frac{d}{dt}\Big|_{t=0} \exp(tX)\cdot z = \frac{d}{dt}\Big|_{t=0} \sum \frac{(tX)^n}{n!} z = Xz.$$

$$\frac{d}{dt}\Big|_{t=0} \exp(tX)\cdot z =: \underline{X}_z.$$

$$\Rightarrow \quad \mathfrak{g} \to \mathfrak{X}(M)$$
$$X \longmapsto \underline{X}$$

$$[\underline{X}, \underline{Y}] = -\underline{[X, Y]}$$

$$(i_{\underline{X}}\omega)_z = dx_i \wedge dy_i(Xz, \cdot)$$

$$\Rightarrow \quad d\tilde{\mu}_X(z) = (i_{\underline{X}}\omega)_z.$$

Identify $\mathfrak{u}(n)$ with $\mathfrak{u}(n)^*$ by

$$A \longmapsto X \mapsto \frac{i}{2}\mathrm{tr}(A^* X)$$

Now $\langle \mu(z), X\rangle = \frac{i}{2}\underline{z^* X} z = \frac{i}{2}\mathrm{Tr}(z^* X z) = \frac{i}{2}\mathrm{Tr}(zz^* \cdot X).$

$$\mu : \mathbb{C}^n \to \mathfrak{u}(n)$$
$$z \longmapsto zz^*$$

Equivariance $\quad Uz \mapsto \underline{Uzz^*U^*}$

(ii) $\mathbb{CP}^{n-1}$

claim : $\omega_{FS} = \omega_{red}$

$$\begin{array}{ccc} S^{2n-1} & \overset{i}{\hookrightarrow} & \mathbb{C}^n \quad \omega_{std} \\ {\scriptstyle \pi}\downarrow & & \\ \mathbb{CP}^{n-1} \quad \omega_{red} & & \end{array}$$

$$\pi^*\omega_{red} = i^*\omega_{std}.$$

ω_{FS}.

$\mathbb{C}P^{n-1}$ $\quad \mathcal{U}_k = \{[z_1, \dots z_n] \mid z_k \neq 0\}$

$\varphi_k : \mathcal{U}_k \to \mathbb{C}^{n-1}$

$$\omega_{FS}|_{\mathcal{U}_k} = \varphi_k^* \left(i \partial \bar\partial \log \frac{\sum |z_i|^2}{|z_k|^2} \right)$$

Fubini-Study form.

Another method to define FS form.

Define 2-form ρ_{FS} $\quad \mathbb{C}^n \backslash \{0\}$ by

$$\rho_{FS} = \frac{i}{2} \partial \bar\partial \log |z|^2 = \frac{i}{2} \left(\frac{dz \wedge d\bar z}{|z|^2} - \frac{\bar z dz \wedge z \cdot d\bar z}{|z|^4} \right)$$

It descends to a symplectic form on $\mathbb{C}P^{n-1}$ denoted by ω_{FS}.

Det $\phi : \mathbb{C}^n \backslash \{0\} \to S^{2n-1}$ by $z \mapsto \frac{z}{|z|}$.

$$\rho_{FS} = \phi^* (\underline{dx_i \wedge dy_i} |_{S^{2n-1}})$$

Now the moment map $\mu : \mathbb{C}P^{n-1} \to \mathfrak{u}(n)$ is

$$[z] \mapsto \frac{z z^*}{\|z\|^2}$$

$$\begin{array}{ccc} \left(\frac{z_1}{\|z\|}, \dots \frac{z_n}{\|z\|} \right) \hookrightarrow & & \overset{\mu}{\to} \\ S^{2n-1} & \hookrightarrow & \mathbb{C}^n \\ \downarrow & & \\ \mathbb{C}P^{n-1} & & \end{array}$$

$[z] = [z_1, \dots z_n]$

(iii) $\underline{su(n) \subset u(n)}$

$\underline{su(n) \hookrightarrow u(n)}$

$$\begin{array}{ccc} & & (A - (\frac{1}{n} \operatorname{tr} A) Id)^* \\ & & \uparrow \\ X \mapsto \frac{i}{2} \operatorname{tr}(A^* X) & & X \mapsto \frac{i}{2} \operatorname{tr}(A^* X) \\ \mathfrak{u}(n)^* & \longrightarrow & su(n)^* \\ \updownarrow & & \updownarrow \\ u(n) & & su(n) \\ \uparrow & & \\ A & & A - (\frac{1}{n} \operatorname{tr} A) Id. \end{array}$$

$$\mu : \mathbb{C}^n \longrightarrow u(n)^* \longrightarrow su(n)^*$$
$$z \longmapsto zz^* \longrightarrow zz^* - \left(\frac{1}{n} tr(zz^*)\right) Id.$$

$$\mu : \mathbb{C}P^{n-1} \to su(n)^*$$
$$[z] \longmapsto \frac{zz^*}{\|z\|^2} - \frac{1}{n} tr\left(\frac{zz^*}{\|z\|^2}\right) Id.$$

(iv) $n=2$

$$S^2 \longrightarrow \mathbb{C}P^1 \xrightarrow{\mu} su(2) = \mathbb{C}\times\mathbb{R}$$
$$(x,y,z) \longmapsto \frac{x+iy}{1-z} = [x+iy, 1-z] \xrightarrow{\mu} \begin{pmatrix} \frac{1}{2}z & \frac{1}{2}x+\frac{1}{2}yi \\ \frac{1}{2}x-\frac{1}{2}yi & -\frac{1}{2}z \end{pmatrix} \mapsto \left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

Ex 9.10. $SO(3)$, $SU(2)$ - action.

Consider the natural action of $SO(3)$ on $S^2 \subset \mathbb{R}^3$

the moment map is inclusion.

$SO(3) \times \underbrace{S^2 \times \cdots \times S^2}_{n \text{ times}} \to S^2 \times \cdots \times S^2$. diagonal action

$\boxed{\text{See [Mcduff] Example 5.3.1}}$

Check that $\underbrace{S^2 \times \cdots \times S^2}_{n \text{ times}} / \mathfrak{S}_n \cong \mathbb{C}P^n$.

Hint. $\mathbb{C}^n / \mathfrak{S}_n \mapsto$ affine space of monic polynomial of degree n

$z_1 \ldots z_n \mapsto \underline{(z-z_1)\cdots(z-z_n)}$

$\underline{d}$

$\underline{\mathbb{C}^n} \; (\underbrace{\_ \; \_ \; \_ \; \cdots \; \_}_{n})$

$S^2 \cong \mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\}$

$z \qquad z$

$$\underline{\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1 / \mathfrak{S}_n} \to \mathbb{C}^{n+1}\setminus\{0\} \xrightarrow{/\mathbb{C}^*} \mathbb{C}P^n$$

$$([x_1,y_1],\cdots [x_n,y_n]) \mapsto \text{coef} \prod_{1\le i\le n} (x_i z - y_i w) \quad (\text{coef}(\quad))$$

$$SO(3) \times (\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1)/\mathfrak{S}_n \qquad \omega_{FS} \times \cdots \times \omega_{FS} = ?$$

$$SO(3) \times \mathbb{C}P^n \to \mathbb{C}P^n$$

Ex 10. $SU(2) \times \mathbb{C}^2 \to \mathbb{C}^2$

hence it acts on

$\mathcal{P} = \{$homogeneous polynomial of degree n in two variables$\}$ of dim $n+1$.

$\forall P \in \mathcal{P}$, $P(z,w)$, $\forall A \in SU(2)$

$$A \cdot P(z,w) = P((z,w)A)$$

Check that the action of $SU(2)$ on $\mathbb{C}P^n$ define this way induces an effective action $SO(3)$.

.. that coincides with the action defined in Ex 9.

pf. $\forall A \in SU(2)$. $A = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$ $|z_1|^2 + |z_2|^2 = 1$.

$$P(z,w) \xmapsto{A} P((z,w)A) = P(z_1 z - \bar{z}_2 w,\ z_2 z + \bar{z}_1 w).$$

$$\mathcal{P}\setminus\{0\}/\mathbb{C}^* \cong \mathbb{C}P^n.$$

$$P \longmapsto \text{coef}(P)$$

Now it remains to prove

$$\bigcap_{x \in \mathbb{C}P^n} \text{stab}(x) = \{\pm I\} \subset SU(2)$$

$\forall A \in \bigcap \mathrm{stab}(x)$ $\quad A = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$

$P(z,w)$ and $P((z,w)A)$ have same solutions for any $P \in \beta$.

$P(z,w) \qquad P(z_1 z - \bar{z}_2 w,\ z_2 z + \bar{z}_1 w)$

$$\begin{cases} z_1 z - \bar{z}_2 w = \lambda z \\ z_2 z + \bar{z}_1 w = \lambda w \end{cases} \quad \text{for some } \lambda \in \mathbb{C}$$

$\Rightarrow z_1 = \lambda,\ \bar{z}_1 = \lambda,\ z_2 = 0. \quad |z_1|^2 + |z_2|^2 = 1. \quad \lambda = \pm 1$

$A = I$ or $-I$.

Coincide with the action defined in Ex 9.

$\prod(x_i z - y_i w) \in \beta. \quad = \prod\left((z,-w)\begin{pmatrix} x_i \\ y_i \end{pmatrix}\right)$

$\forall A \in SO(3) \quad$ Ex 9 $\longrightarrow \prod\left((z,-w)\, A\begin{pmatrix} x_i \\ y_i \end{pmatrix}\right)$

Ex 10 $\longrightarrow \prod\left((z,-w)A \begin{pmatrix} x_i \\ y_i \end{pmatrix}\right)$

Ex 14. $SO(n+1) \times \mathbb{C}^{n+1}$ (diagonally)

$A \cdot (x+iy) = Ax + iAy.$

Identifying $so(n+1)^*$ with the vector space $\Lambda^2 \mathbb{R}^{n+1}$

(i) prove that the moment map is

$$x + iy \mapsto x \wedge y \qquad x, y \in \mathbb{R}^{n+1}.$$

(ii) Deduce a Hamiltonian action of $SO(n+1)$ on $\mathbb{C}P^n$

$$[x+iy] \mapsto \frac{x \wedge y}{\|x\|^2 + \|y\|^2}$$

(iii) Prove $\|\mu\|^2$ is maximal along the submanifold Q of $\mathbb{C}P^n$

$$Q = \left\{ \sum_{i=0}^{n} z_i^2 = 0 \right\}$$

and $Q \cong \widetilde{G}_2(\mathbb{R}^{n+1})$.

(iv) Check that the complete of Q in $\mathbb{C}P^n$ is diffeomorphism with an open disc bundle in $T(\mathbb{R}P^n)$.

(Eg of Lagrangian barrier)

P. Biran

pf. (i) $so(n+1)^*$ $\quad\quad$ $\Lambda^2(\mathbb{R}^{n+1})$

$A = (a_{ij}) \longleftrightarrow (x,y) \mapsto x^T A y \longleftrightarrow (a_{ij} - a_{ji})\, e_i^* \wedge e_j^*$

$$\begin{tikzcd}
so(n+1) \arrow[r, leftrightarrow] \arrow[d, dash] \arrow[dr, phantom, "\hookrightarrow"] & \Lambda^2(\mathbb{R}^{n+1})^\vee \arrow[d, leftrightarrow] & e_i^* \wedge e_j^* \ (i<j) \arrow[d, leftrightarrow] \\
so(n+1)^* \arrow[r, leftrightarrow] & \Lambda^2(\mathbb{R}^{n+1}) & e_i \wedge e_j \ (i<j)
\end{tikzcd}$$

Plücker.

$x \wedge y \cdot (A) = x A y$

$\mathbb{C}^{n+1} \to so(n+1)^*$

$\mu: x + iy \mapsto x \wedge y$

$\forall A \in so(n+1)$, $A + A^T = 0$.

$$\tilde{\mu}_A(x+iy) = (x \wedge y) \cdot A = x^T A y = x_i a_{ij} y_j.$$

$$d\tilde{\mu}_A(x+iy) = a_{ij} y_j \, dx_i + x_i a_{ij} \, dy_j.$$

Similarly. $\underline{X}_{x+iy} = A(x+iy) = Ax + iAy.$

$\omega = dx_i \wedge dy_i$

$i_{\underline{X}} \omega = dx_i \wedge dy_i (Ax + iAy , \cdot)$

$\Rightarrow d\tilde{\mu}_A (x+iy) = (i_{\underline{X}} \omega)_{x+iy}$

$\mu(x+iy) = x \wedge y$ $\qquad B \mapsto x^T B y$

$\mu(\underline{Ax+iAy}) = \underline{Ax \wedge Ay}$ $\qquad B \mapsto x^T \underline{A^T B A} y.$

μ is equivariant.

(ii) $\mu : \mathbb{C}^{n+1} \to so(n+1)^*$

$x+iy \mapsto x \wedge y.$

$\mu : \mathbb{CP}^n \to so(n+1)^*$

$[x+iy] \mapsto \dfrac{x \wedge y}{\|x\|^2 + \|y\|^2}. \in \wedge^2(\mathbb{R}^{n+1})$

$e_i \wedge e_j \ (i<j)$

(iii) $\mu = \dfrac{x \wedge y}{\|x\|^2+\|y\|^2} = \dfrac{(x_i y_j - x_j y_i) e_i \wedge e_j}{\|x\|^2 + \|y\|^2}$

$\|\mu\|^2 = \dfrac{\Sigma (x_i y_j - x_j y_i)^2}{(\|x\|^2 + \|y\|^2)^2}$ $\qquad \{\Sigma z_i^2 = 0\}$

$\Updownarrow$

$= \dfrac{\Sigma x_i^2 y_j^2 + \Sigma x_j^2 y_i^2 - 2 \Sigma x_i x_j y_i y_j}{(\qquad)^2}$ $\qquad \begin{cases} \Sigma x_i^2 = \Sigma y_i^2 \\ \Sigma x_i y_i = 0. \end{cases}$

$= \dfrac{2(\Sigma x_i^2)(\Sigma y_i^2) - 2(\Sigma x_i y_i)^2}{(\Sigma x_i^2 + \Sigma y_i^2)^2}$

$\leq 2 \dfrac{\Sigma x_i^2 \Sigma y_i^2}{(\Sigma x_i^2 + \Sigma y_i^2)^2}$ $\qquad$ "=" iff $\Sigma x_i y_i = 0.$

$$\leq 2 \cdot \frac{1}{4} \qquad \text{"="} \text{ iff } \sum x_i^2 = \sum y_i^2$$

$\|\mu\|^2$ maximal. $Q = \{\sum z_i^2 = 0\}$.

$Q \cong \widetilde{G}_2(\mathbb{R}^{n+1})$

$[z_0, \cdots z_n] \mapsto \mathrm{span}\{x, y\}$ $\qquad \begin{cases} \sum x_i^2 = \sum y_i^2 \\ \sum x_i y_i = 0 \end{cases}$

$x = (x_0, \cdots x_n)$
$y = (y_0, \cdots y_n)$ $\in \mathbb{R}^{n+1}$

Ex 19. (Symplectic cutting)

H: period Hamiltonian on symplectic manifold W.

consider S^1-action on $W \times \mathbb{C}$ by

$$u \cdot (x, z) = (u \cdot x, \bar{u} z)$$

and moment map $\tilde{H}(x, z) = H(x) - \frac{1}{2}|z|^2$

Assume $t > 0$ (See P195 of [Audin])

and S^1-action on the level $H^{-1}(t)$ is free

then t is a regular value of $\tilde{H}$.

$$\tilde{H}^{-1}(t) = \{(x, 0) \mid H(x) = t\} \sqcup \{(x, z) \mid H(x) > t,\ \underline{|z|^2 = 2(H(x) - t)}\}$$

$$H^{-1}(t) \qquad \sqcup \qquad H^{-1}((t, +\infty)) \times S^1$$

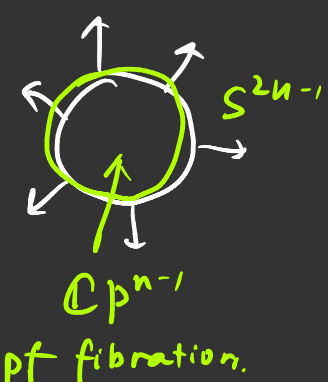
both are S^1-invariant.

$$\widetilde{H}^{-1}(t)/S^1 = \underbrace{H^{-1}(t)/S^1}_{\text{previous reduced space}} \amalg \underbrace{H^{-1}((t,+\infty))}_{\text{upper level set}}$$

We have cut W at the level t of H and "closed" it by $H^{-1}(t)/S^1$

Ex. (blow up). $W = \mathbb{C}^n$ $\quad H(x) = \frac{1}{2}\|x\|^2$ $\quad (t>0)$

$$\underbrace{H^{-1}(t)/S^1}_{\mathbb{C}P^{n-1}} \amalg \underbrace{H^{-1}((t,+\infty))}_{\|x\|^2 > 2t}$$



This is the symplectic blow up.

The complex blow up of $\mathbb{C}^n$ at the origin is the complex mfd

$$\widetilde{\mathbb{C}^n} = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}P^{n-1} \mid z_i w_j = z_j w_i \ \forall i,j\}$$

$\pi: \widetilde{\mathbb{C}^n} \to \mathbb{C}^n$ $\qquad pr: \widetilde{\mathbb{C}^n} \to \mathbb{C}P^{n-1}$.

Away from $\pi^{-1}(0)$, π is a biholomorphism

$$\pi^{-1}(0) = \mathbb{C}P^{n-1}.$$

$$pr^{-1}([w_1, \dots w_n]) = (\lambda(w_1,\dots,w_n), [w_1,\dots w_n])$$

$\widetilde{\mathbb{C}^n}$ $\quad$ $\mathbb{C}P^{n-1}$: tautological line bundle.

$\tilde{\mathbb{C}}^n$ is symplectic mfd. $(\lambda > 0)$

$$\omega_\lambda = \pi^* \omega_{std} + \lambda^2 pr^* \omega_{FS}.$$

([McDuff] Sec 7.1)

$\tilde{\mathbb{C}}^n \qquad \mathbb{C}^n \setminus \{0\} \qquad \mathbb{C}P^{n-1}$

$\tilde{H}^{-1}(t)/S^1 \qquad \mathbb{C}^n \setminus B_{\sqrt{2t}} \qquad \mathbb{C}P^{n-1}$

lemma. $\forall \lambda > 0 \quad \forall \delta \in (0, \infty]$, the map

$$F : (\pi^{-1}(B_\delta \setminus \{0\}), \omega_\lambda) \to (B_{\sqrt{\delta^2+\lambda^2}} \setminus B_\lambda, \omega_{std})$$

$$z \mapsto \sqrt{|z|^2 + \lambda^2}\,\frac{z}{|z|}.$$

is a symplectomorphism.

$$\omega_\lambda = \frac{i}{2}\partial\bar{\partial}(\lambda^2 \log |z|^2 + |z|^2)$$

$$\omega_{std} = dx_i \wedge dy_i$$

$t > 0$. P195. $\underline{\tilde{H}^{-1}(x,t)/S^1}$ blow up

$\underline{t = 0 \text{ critical point}}$

$t < 0 \qquad \tilde{H}^{-1}(x,t)/S^1 \qquad \underline{H^{-1}(t)}$