Lecture Notes

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In our previously courses, we know that in the case of Hamiltonian $\mathbb{S}^1$ actions on a connected compact symplectic manifold M[1], the critical points of the moment map $\mu : M \longrightarrow \mathbb{R} \cong (\mathfrak{s}^1)^*$ are exactly the fixed points of the action, in the case of the torus action $\mathbb{T} \times M \longrightarrow M$, we wish to construct a *single* function f, such that the critical points of which are the fixed points of this torus action, so that we can use the Morse theory to study the action.

It is not difficult to find such a function, we can choose a vector $X \in \mathfrak{t}$, such that the one parameter subgroup $H = \{\exp tX : t \in \mathbb{R}\}$ is dense in $\mathbb{T}$[2],then we consider the composition

$$M \xrightarrow{\mu} \mathfrak{t}^* \xrightarrow{i^*} \mathfrak{h}^* \cong \mathbb{R}$$

$$f(x) = \langle \mu(x), X \rangle = \mu_X(x)$$

[3] Then, by the construction, this function f has the critical points as the fixed points of the torus action.

With a further observation, the associated Hamiltonian vector field $X_f(x) = \underline{X}(x)$, which is exactly the fundamental vector field of $X \in \mathfrak{t}$, the flow of X_f is equivalent to the flow of X on that torus $\mathbb{T}$, thus the closure of the one parameter group

$$\overline{L} = \overline{\left\{ \phi^t_{X_f} : t \in \mathbb{R} \right\}}$$

is diffeomorphic to the torus $\mathbb{T}$ as the subgroup of $\mathrm{Diff}(M)$. Such functions will later be called the almost periodic Hamiltonians.

1 Almost Periodic Hamiltonians

From our previously lectures, we defined the **periodic Hamiltonians**[4], which is a function $H : M \longrightarrow \mathbb{R}$ such that for every $p \in H^{-1}(x)$, the associated Hamiltonian vector field X_H at p lies in the tangent space of the level set $H^{-1}(x)$, i.e. $X_H(p) \in T_p(H^{-1}(x))$, for all $H(p) = x$.

Definition 1. *A function $H : M \longrightarrow \mathbb{R}$ is said to be an **almost periodic Hamiltonian**, if the closure of the subgroup:*

$$\overline{L} = \overline{\{\phi^t : M \longrightarrow M | t \in \mathbb{R}\}} \subset \mathrm{Diff}(M)$$

is a torus, where ϕ^t[5] is the flow induced by the Hamiltonian vector field X_H associated to H.

We recall that the one-parameter group $\left\{ \phi^t_X : t \in \mathbb{R} \right\}$ generated by the flow of any vector field X is a connected Abelian group[6], thus the only assumption here is to assume that $\overline{L}$ is a compact one.

Note that the flow of a periodic hamiltonian vector field is equivalent to an $\mathbb{S}^1$ action, hence we have:

Proposition 1. *A periodic Hamiltonian is an almost periodic Hamiltonian.*

Although the almost periodicity is the generalization of the periodicity, but the assumption is not mild, it may have a difference between H and its H^2.

Example: Consider $\mathbb{R}^2$ with the usual symplectic form $\omega = \mathrm{d}p \wedge \mathrm{d}q$, let

$$H = \frac{1}{2}\left(p^2 + q^2\right) : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

[1]In this lecture, we assume M is a **compact connected** symplectic manifold. If without additional explanations, all symplectic manifolds will be connected and compact.

[2]This is not hard to make it, for example, $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, we choose $X \in \mathbb{R}^2 \cong \mathfrak{t}^2$, such that the slope of X is irrational, then its flow line is dense in $\mathbb{T}^2$.

[3]Is is right in this case the pairing $\langle \mu(x), X \rangle$ is the inner product?

[4]Why would we call it **periodic**? Does it have some relations with the periodicity of an ODE system? Yes, the Hamiltonian vector field X_H will define an ODE system, that is

$$\gamma'(t) = X_H \circ \gamma(t)$$

H is periodic implies that system has periodic solutions, that is because a solution of which is closed 1-dimensional submanifold, notice that $X_H(p) \in T_p(H^{-1}(x))$ implies the trajectories of which are lying in $H^{-1}(x)$, which is a compact submanifold, thus the solution line of which is a compact 1-dimensional manifold, that is $\mathbb{S}^1$, which is exactly a periodic solution.

[5]Here we use ϕ^t_X to express a flow induced by a vector field X, and $\exp tX$ for the trajectory of X, obviously, the flow ϕ^t acts by moving a point $p \in M$ along the trajectory, which passed through p at the origin, $t-$times after.

[6]As for the commutativity, moving a point $p \in M$ along the trajectory by $s-$times first and $t-$ times after is equivalent to by moving $t-$times first and $s-$times after, which are equivalent to by moving $(s+t)-$times, as for the connectedness, it is easy to find a path which connects any two flows, namely $\phi^{t_1}_X, \phi^{t_2}_X$, via

$$\gamma(s) = \phi_X^{st_1 + (1-s)t_2}$$

thanks to the compactness of M.

the associated Hamiltonian vector field X_H is by sending $z \in \mathbb{R}$ to the tangent vector of the circle at z which centered at the origin, it is obviously periodic hence almost periodic, consider its square H^2, it is not hard to find the flow of X_{H^2} is

$$\phi^t_{X_{H^2}}(z) = e^{i|z|^2 t}.z$$

however, the closure of which is not a compact one. Indeed, we denoted by

$$L = \left\{\phi^t_{X_{H^2}} : z \to e^{i|z|^2 t}.z | t \in \mathbb{R}\right\}$$

Recall that the topology on $\mathrm{Diff}(\mathbb{R}^2)$ is compact-open topology, thus the evaluation map

$$\psi : \overline{L} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

is continuous. Now, if $\overline{L}$ is compact, choose $z \in \mathbb{R}^2$ so that $|z|^2 \in \mathbb{R} \setminus \mathbb{Q}$, its image under ψ is also compact, since $\mathbb{R}^2$ is metric space, thus the image will be sequential compact, choose a sequence $\{e^{i|z|^2 nt}.z\}_{n \geqslant 0}$, thus any subsequence of which is a dense subset of $\mathbb{S}^1$ hence not convergent, a desired contradiction. ■

By a simple observation, we have :

Proposition 2. *Any almost periodic Hamiltonian H is a paring of the moment map of a torus action, i.e. there exists a torus action $\mathbb{T} \times M \longrightarrow M$, and a vector $X \in \mathfrak{t}$, such that $H(x) = \langle \mu(x), X \rangle$, where μ is the moment map. Hence then the zeros of X_H will be the fixed points of that $\mathbb{T}$ action.*

Actually, this torus is $\bar{L}$ with the usual action, that $\mathbb{T}$ will be called the torus generated by H.

Example: For the case of the torus action on $\mathbb{CP}^n$:

$$\mathbb{T}^n = \left\{(t_0, ..., t_n) \in \mathbb{T}^{n+1} | t_0 ... t_n = 1\right\}$$

via

$$(t_0, ..., t_n).[z_0, ..., z_n] \mapsto [t_0 z_0, ..., t_n z_n]$$

As we have computed before, this is a Hamiltonian action with the moment map

$$\mu([z_0, ..., z_n]) = \frac{1}{2}\left(\frac{|z_0|}{\sum_{i=0}^n |z_i|^2}, ..., \frac{|z_n|}{\sum_{i=0}^n |z_i|^2}\right)$$

Now if we fix some proper real numbers $a_0, ..., a_n$, we will have an almost periodic Hamiltonian:

$$H([z_0, ..., z_n]) = \frac{1}{2}\frac{\sum_{i=0}^n a_i |z_i|^2}{\sum_{i=0}^n |z_i|^2}$$

As we have proven in the Morse theory, this function is a Morse function of $\mathbb{CP}^n$, with critical point $[0, ..., 1, ..., 0]$, which are exactly the fixed points of that torus action.

2 Second Derivative of H

Let H be an almost periodic Hamiltonian, and $\mathbb{T}$ the torus generated by H, we denoted by Z the zeros of X_H, which are the fixed points of this $\mathbb{T}$ action, recall that

$$\mathbb{T} = \overline{\{\exp tX_H | t \in \mathbb{R}\}}$$

For any $s \in \mathbb{T}$, and any $z \in Z$, s induces an isomorphism of the tangent space:

$$(\mathrm{d}s)_z : T_z M \longrightarrow T_z M$$

Note that there is an almost complex structure J calibrated with the symplectic form ω on $T_z M$, thus has an Hermitian form on $T_z M$, since the torus action is Hamiltonian, it preserves the Hermitian structure, hence the torus $\mathbb{T}$ can be regarded as the subgroup of $\mathrm{U}(n)$, thus its Lie algebra $\mathfrak{t} \subset \mathfrak{u}(n)$, since $\mathbb{T}$ is Abelian, thus all elements can be simultaneously diagonalized, i.e. they have common eigenvectors, moreover, since $\mathbb{T}$ fixes Z, thus $T_z Z$ is spanned by the eigenvectors subordinate to the eigenvalue 1, thus we can write

$$T_z M = \bigoplus_{i=0}^k V_i$$

where $V_0 = T_z Z$, and the rests V_i are **complex** 1-dimensional.

We suppose $v_0^1, ..., v_0^r, v_1, ..., v_k$ is a basis of T_zM, now let's see how $\exp X_H$ acts on these decomposition. Recall that the eigenvalues of an unitary matrix are unimodulars, thus $(\exp X_H)v_j = e^{i\lambda_j}v_j$ for $j = 1, ..., k$, and $(\exp X_H)v_0^i = v_0^i$, thus the matrix of X_H is

$$\begin{pmatrix} O_{r\times r} & & & \\ & i\lambda_1 & & \\ & & \ddots & \\ & & & i\lambda_k \end{pmatrix}$$

we recall that $\mathfrak{t}$ has an infinitesimal version of the torus action on T_zM by

$$\begin{aligned} X_H(v) = \overline{X_H}(v) &= \left.\frac{\mathrm{d}(\mathrm{d}\exp tX_H)_z v}{\mathrm{d}t}\right|_{t=0} \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\left.\frac{\mathrm{d}(\exp tX_H)\gamma(s)}{\mathrm{d}s}\right|_{s=0} \right) \\ &= \nabla_{X_H}\gamma'(0) \end{aligned}$$

where $\gamma(0) = z$ and $\gamma'(0) = v$, and ∇ is the Levi-Civita connection of the $\omega-$tamed Riemann metric, hence the matrix of X_H is the matrix of the linear transformation

$$\nabla_{X_H} : T_zM \longrightarrow T_zM$$

Also recall that the Hessian of H at point z under this basis is the bilinear form:

$$(\nabla^2 H)_z : T_zM \times T_zM \longrightarrow \mathbb{R}$$

the matrix of which is the matrix of the linear transformation

$$\nabla_{\mathrm{grad}H} = -J\nabla_{X_H} : T_zM \longrightarrow T_zM$$

hence which is

$$\begin{pmatrix} O_{r\times r} & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_k \end{pmatrix}$$

thus the Hessian is the quadratic form

$$\sum_{i=1}^{k} \lambda_i |v_i|^2$$

So now, we can define what is the Morse function.

Definition 2 (Morse function in the sense of Bott)**.** *A function $f : M \longrightarrow \mathbb{R}$ is a Morse function if*

1). The critical points of which is a submanifold.

2). The Hessian of f at any critical points is non-degenerate along the transverse direction.

As we can see, for an almost periodic Hamiltonian H it is a Morse function, and thanks to the complex vectors, the index of any critical points must be even.

Theorem 1 (Frankel)**.** *The almost periodic Hamiltonian function is a Morse function, and all the critical points are of even index.*