Delzant Thm.

object

Definition (symplectic toric 流形): → $\dim(M)=2n$.

紧, 连通, 辛流形. (M,ω,T^n,μ).

T^n-作用有效 Hamiton

$\boxed{\mathbb{C}^N/(\mathbb{C}^*)^n}$

Gauged linear σ-model

$(t^n)^*$

凸 顶点

$\sum_{i=1}^{n} p_i s_i w_i$, $w_i\in\mathbb{Z}_i$ weight.

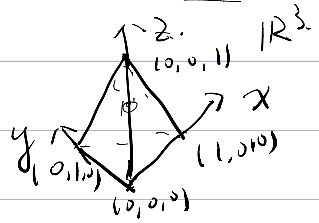
Definition (Delzant polytope):

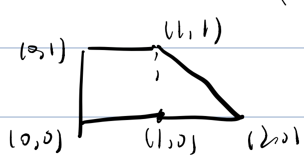
多面体 $\Delta\subseteq\mathbb{R}^n$ 称为 Delzant polytope.

若: (1) 简单 $\forall$顶点 p 连有 n 个边

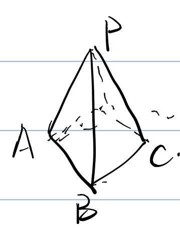
(2) 有理 与 p 点相交的棱: $p+tu_i$, $u_i\in\mathbb{Z}^n$.

fibration (3) 光滑 $\forall p$. $u_1,\dots,u_n$ 张成 $\mathbb{Z}^n$ 中一组基

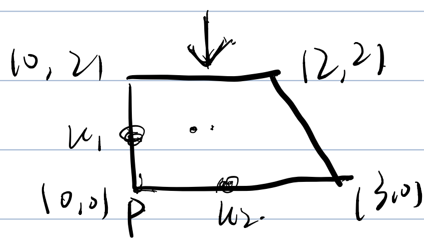




← Delzant polytope.



(不) simplex.



(光滑)

$\boxed{\begin{vmatrix}0&2\\3&0\end{vmatrix}}$ $\mathbb{Z}\times\mathbb{Z}\to 2\mathbb{Z}\times3\mathbb{Z}$

$\mathbb{Z}$-矩阵 可逆

$\Updownarrow$

$\mathrm{span}\{u_1,\dots,u_n\}=\mathbb{Z}^n$

Delzant Thm.

symplectic toric mfd (等变辛微分同胚)

moment map $\Updownarrow$ 1-1.

Delzant polytope $\triangle$

$\boxed{\text{well-defined}}$

proof: Step 1. symplectic toric → Delzant polytope.

(M,ω,T^n,μ).

z 是 T^n 作用不动点，设 $\mu(z)=p$，$\exists\, U\subseteq M$，$z\in U$.

$\mu(U)=p+\sum_{i=1}^{n} s_i w_i$　　$w_i\in\mathbb{Z}^n$.

光滑 ——

(反证) 若 w_i 不张成 $\mathbb{Z}^n$，$W=(w_i)_{1\le i\le n}$ 关于 $\mathbb{Z}$ 不可逆

① 关于 $\mathbb{R}$ 不可逆 $\boxed{\text{显然}}$ $\exists\, X\in\mathbb{R}^n/\mathbb{Z}^n$, $W(X)=0$

② 关于 $\mathbb{R}$ 可逆，但关于 $\mathbb{Z}$ 不可逆，$\exists\, X\in\mathbb{R}^n/\mathbb{Z}^n$. $W(X)\in\mathbb{Z}$.

$\Downarrow$

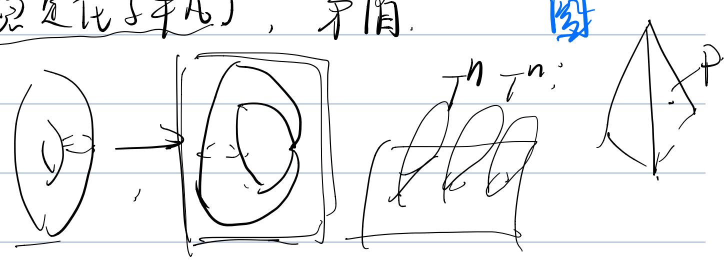
$\exists\, X\in\mathbb{R}^n/\mathbb{Z}^n$. $W(X)\in\mathbb{Z}$.

$w_i\in\mathfrak{t}^*$　$X\in\mathfrak{t}$　$\langle w_i,X\rangle\in\mathbb{Z}$　$(z_1,\dots,z_n)\in U$

$$\underset{T^n}{\exp(X)\curvearrowright}\cdot(z_1,\dots,z_n)=(e^{2\pi i\langle w_1,X\rangle}z_1,\dots,e^{2\pi i\langle w_n,X\rangle}z_n)$$
$$=(z_1,\dots,z_n)$$

$\exp(X)$ 是 U 上非平凡稳定化子.

(有效 ⇒ 主轨道上稳定化子平凡)，矛盾.　　图

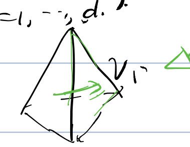


Step 2.

Delzant polytope Δ → symplectic toric M_Δ　设 Δ 有 d 个面. $d\ge n$.

$\Delta=\{\xi\in\mathbb{R}^n \mid \langle\xi, v_i\rangle\le\lambda_i,\ v_i\in\mathbb{R}^n,\ \lambda_i\in\mathbb{R}\}$　$i=1,\dots,d$.

$\mathfrak{t}^*$　　(法向朝外), primitive



$T^d\times\mathbb{C}^d\to\mathbb{C}^d$

$(e^{i\theta_1},\dots,e^{i\theta_d})(z_1,\dots,z_d)\to(e^{i\theta_1}z_1,\dots,e^{i\theta_d}z_d)$

$\mu(z_1,\dots,z_d)=\left(-\frac{1}{2}(|z_1|^2,\dots,|z_d|^2)+\lambda\right)$　　$\lambda=(\lambda_1,\dots,\lambda_d)$

有效 — Hamilton 作用 $\qquad \operatorname{Im}\mu = (t^d)^* = \mathbb{R}^d$

K-线性: $\mathbb{R}^d \to \mathbb{R}^n = (t^n)^*$ $\qquad i=1,\dots,d.$

$e_i \to v_i$

$\pi: T^d \to T^n \qquad \exp(\mathbb{R}^d) \to \exp(\mathbb{R}^n)$

记 $N = \ker\pi = T^d/T^n \cong T^{d-n}$.

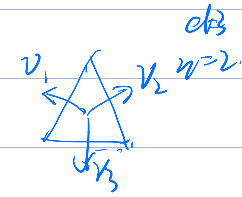
$$0 \to N \xrightarrow{i} T^d \xrightarrow{\pi} T^n \to 0$$

↓ 李代数 T_e

$$0 \to n \xrightarrow{i} \underset{t^d}{\mathbb{R}^d} \xrightarrow{\pi} \underset{t^n}{\mathbb{R}^n} \to 0$$

↓ 李代数对偶

$$0 \to \underset{(t^n)^*}{\mathbb{R}^n} \xrightarrow{\pi^*} \underset{(t^d)^*}{\mathbb{R}^d} \xrightarrow{i^*} n^* \to 0$$



$T^d \times \mathbb{C}^d \to \mathbb{C}^d$ Hamilton 作用 μ

要证 $N \times \mathbb{C}^d \to \mathbb{C}^d$ Hamilton 作用, $\mu' = i^* \circ \mu$

$M_\Delta = \underbrace{\ker\mu'}_{= Z} / N$ — 紧, 连通, 辛流形 $\}$ symplectic toric

T^n-有效 Hamilton

且 $\mu(M_\Delta) = \Delta$

Step 2.1. ① N 在 Z 上自由作用 ② Z 紧 $\}\Rightarrow M_\Delta$ 紧, 辛

$$\begin{tikzcd} \mathbb{C}^d \arrow[r, "\mu"] \arrow[dr, "\mu'"'] & (t^d)^* = \mathbb{R}^d \arrow[d, "i^*"] \\ & n^* \end{tikzcd}$$

prof: (Z 紧)

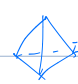
$Z = \ker\mu'$

$= \ker(i^* \circ \mu)$

$= (i^* \circ \mu)^{-1}(0)$

$= \mu^{-1}(\ker i^*) = \mu^{-1}(\operatorname{Im}\pi^*) = \mu^{-1}(\Delta')$ $\quad$ ∵ μ proper

要证 $\operatorname{Im}\pi^*$ 紧



记 $\Delta' = \pi^*(\Delta) \subseteq \mathbb{R}^d \Rightarrow \Delta'$ 紧

claim: $\Delta' = \operatorname{Im}(\pi^*) \cap \operatorname{Im}(\mu)$.

$$\Delta' = \pi^*(\Delta)$$

$$= \{\pi^* x \mid x \in (t^n)^*,\ \langle x, \overset{\pi(e_i)}{\overset{\parallel}{v_i}} \rangle \le \lambda_i \}$$

$$= \{\pi^* x \mid \langle \pi^* x, e_i \rangle \le \lambda_i \}$$

$$= \mathrm{Im}(\mu) \cap \mathrm{Im}(\pi^*)$$

proof of

① N 在 $Z = \ker\mu'$ 上作用自由

$\Updownarrow$

proof: $N_z = \{0\}$ $\forall z \in \ker\mu'$.

$N \xrightarrow{i} T^d \xrightarrow{\psi} \mathrm{Diff}(\mathbb{C}^d)$

$N_z = i(N) \cap T^d_z$

$\pi: T^d \to T^n$

$= \ker(\pi) \cap T^d_z$.

$(e_1, \cdots, e_d) \to (v_1, \cdots, v_d)$

$= \ker(\pi|_{T^d_z}) \overset{?}{=} \{0\}$

claim: $\pi|_{T^d_z}: T^d_z \to T^n$ 单

proof of claim: 单 $\Leftrightarrow$ $\forall\, t, s \in T^d_z$ $t = \sum_{i=1}^d t_i e_i$ $s = \sum_{i=1}^d s_i e_i$

$$\pi(t) = \sum_{i=1}^d t_i v_i$$

$$\pi(s) = \sum_{i=1}^d s_i v_i$$

若 $\pi(t) - \pi(s) = \boxed{\sum_{i=1}^d (t_i - s_i) v_i = 0} \overset{?}{\Rightarrow} t_i = s_i$

($\because$ v_i 线性相关，以下取 v_i 线性无关的指标)

记 $I(z) = \{ i \in I \mid z_i = 0 \}$. (零系数坐标)

$\boxed{(T^d)_z = \{ t \in T^d \mid t_i = 0,\ i \notin I_z \}}$

T^d 在 $\mathbb{C}^d$ 上作用 $(\underset{t_1}{e^{i\theta_1}}, \cdots, \underset{t_d}{e^{i\theta_d}})(z_1, \cdots, z_d) = (\underset{t_1}{e^{i\theta_1} z_1}, \cdots, \underset{t_d}{e^{i\theta_d} z_d})$

若 $i \in I_z$, $z_i = 0$, t_i 取全部 S^1

$i \notin I_z$, $z_i \ne 0$, $t_i = \{0\}$.

$\therefore$ 只有 I_z 中指标 $t_i \ne 0$

$\Rightarrow \pi(t) - \pi(s) = \sum_{i \in I_z} (t_i - s_i) v_i$ $t_i = s_i$ $\Rightarrow \boxed{t = s}$ $(T^d)_z \to T^n$ 单

图

claim: $i \in I_z$, $\{v_i\}_{i \in I_z}$ 线性无关.

$\therefore \mathbb{C}^d \to (t^d)^* = \mathbb{R}^d$

proof of claim: $i \in I_z \Leftrightarrow z_i = 0$ $\qquad (z_1, \cdots, z_d) \to -\frac{1}{2}(|z_1|^2, \cdots, |z_d|^2) + \lambda$

$\because z \in \ker \mu'$, 于 $\mu' = i^* \mu$ $\quad \Leftrightarrow \langle \mu(z), e_i \rangle = \lambda_i$

$\exists x \in \mathbb{R}^n$, $\pi^*(x) = \mu(z)$ $\quad \Leftrightarrow \langle \pi^*(x), \underset{\pi(e_i)}{e_i} \rangle = \lambda_i$

$\Leftrightarrow \langle x, v_i \rangle = \lambda_i$

$\Leftrightarrow$ x 在 v_i 决定的超平面交点里

$\therefore v_i$ 线性无关 $\qquad \blacksquare$

Step 2.2. 有效 T^n-Hamilton 作用 $\qquad T^d \times \mathbb{C}^d \to \mathbb{C}^d$

找合适嵌入. $\qquad N \times \mathbb{C}^d \to \mathbb{C}^d$

$\tilde{j}: T^n \hookrightarrow T^d$ $\quad \tilde{j}$ 已知 $\pi: (T^d)_z \to T^n$ 单射)

下证 $\pi|_{(T^d)_z}: (T^d)_z \to T^n$ 双射

即证 $\dim (T^d)_z = n$.

proof: $\forall t \in (T^d)_z$. $\pi(t) = \sum_{i \in I_z} t_i v_i$

(v_i 是包含 $\mu(z) = \pi^*(x)$ 中 x 的线性无关的面坐标)

$\because \Delta$ 是 Delzant. x 有 n 个面.

$\therefore |I_z| = n$

$\Rightarrow \dim (T^d)_z = n$.

$\Rightarrow \tilde{j}: T^n \to T^d$ 是嵌入 $\pi \tilde{j} = Id$.

$$\begin{tikzcd} \mathbb{C}^d \arrow[r, "\mu"] \arrow[dr, "\mu'"'] & (t^d)^* = \mathbb{R}^d \arrow[d, "\tilde{j}^*"] \\ & (t^n)^* \end{tikzcd}$$

那么由 $T^d \times \mathbb{C}^d \to \mathbb{C}^d$ 有效 Hamilton 作用

诱导 $T^n \times \mathbb{C}^d \to \mathbb{C}^d$ 有效 Hamilton 作用 $\mu' = \tilde{j}^* \circ \mu$

$N \to \mathrm{Diff}(\mathbb{C}^d)$ 由 T^d 诱导

$T^n \to \mathrm{Diff}(\mathbb{C}^d)$

$\because$ N 作用和 T^n 作用交换

proposition $\Rightarrow$ T^n 在 Z/N 上是哈密顿作用

$$\underbrace{\eta_{red}}_{\tilde{\mu}''} \circ Pr = \mu'' \circ j$$

$i : Z \to \mathbb{C}^d$

$Pr : Z \to Z/N$

$$\begin{tikzcd}
Z \arrow[r, "j"] \arrow[d, "Pr"'] & \mathbb{C}^d \arrow[d, "\tilde{j}^* \circ \mu = \mu''"] \\
\underset{M_\Delta''}{Z/N} \arrow[r, "\tilde{\mu}''"] & (t^n)^*
\end{tikzcd}$$

Step 2.3. $\tilde{\mu}''(M_\Delta) = \Delta$

proof: $\tilde{\mu}''(M_\Delta)$

$= \tilde{\mu}'' \circ Pr(Z)$

$= \tilde{j}^* \circ \mu \circ j(Z)$

$= \tilde{j}^* \circ \mu(\ker(i^* \circ \mu))$

$= \tilde{j}^* \circ \mu(\mu^{-1}(\ker i^*))$

$= \tilde{j}^*(\ker i^*)$

$= \tilde{j}^*(\mathrm{Im}(\pi^*)) = \tilde{j}^*(\pi^*(\Delta)) = (\pi \circ \tilde{j})^*(\Delta)$

$= \Delta$

Delzant polytope M_Δ

$\Rightarrow \Delta \longrightarrow$ symplectic toric