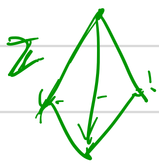
Delzant Thm.

symplectic toric Manifold (辛等变微分 ≅)

$\Updownarrow$ 1-1, μ. 　紧. 连通. 辛. 有效 T^n-Hamilton

Delzant polytope.



存在 v_i 　$v_i \in \mathbb{Z}^n$

$\{v_i\}$ = $\mathbb{Z}^n$ 中一组基.

1. sym toric $\xrightarrow{\mu}$ Delzant polytope. (well defined)
2. Delzant polytope $\xrightarrow{\text{construct}}$ symplectic toric (满)
3. 单?

Thm. (M,ω), (M',ω') 2n维紧连通辛流形

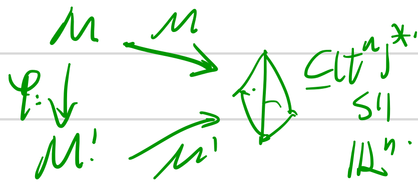
T^n 有效作用在 M, M' 上.

$\mu: M \to \mathbb{R}^n$ 　　且 $\mu(M) = \mu'(M') = \Delta$

$\mu': M' \to \mathbb{R}^n$

那么 ∃ 等变辛微分同胚 $\varphi: M \to M'$

$\mu = \mu' \circ \varphi$



idea: ① $\forall p \in \Delta$. $\exists \tilde{U} \ni p$. 　$\tilde{U} \subseteq \mathbb{R}^n$.

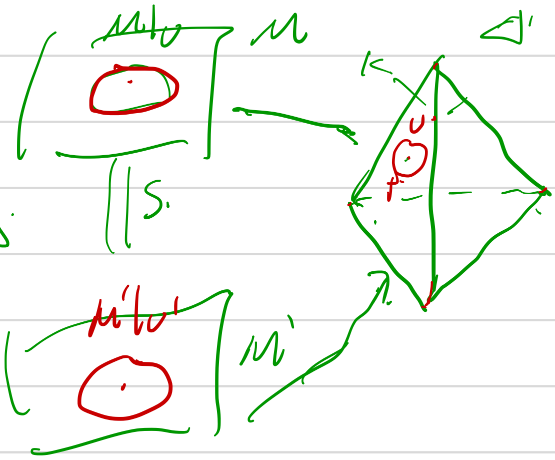
∃ $M|_U$, $M'|_{U'}$.

$\mu(M|_U) = \mu'(M'|_{U'}) = U \cap \Delta$

那么 $M|_U \cong M'|_{U'}$ (等变微分)

② ⇒ ∃ f: $M \to M'$ 等变微分

③ f → 辛等变 diff. ✓



(idea ③)

Lemma: (M,ω,T^n,μ), (M',ω',T^n,μ') 辛 toric (2n)

$\mu(M)=\mu(M')=\Delta$ ①②

若 $\boxed{f: M\to M' \text{ 等变微分同胚}}$ $\mu'\circ f=\mu$. $(M, f^*\omega)$

记 $\omega_0=\omega$ $\omega_1=f^*\omega'$

$\omega_t=(1-t)\omega_0+t\omega_1$

$\Rightarrow$ ∃ 等变微分同胚

$\rho_t: M\to M$

s.t. $\boxed{\mu\circ\rho_t=\mu}$. $\rho_t^*\omega_t=\omega_0$.

claim 3.

$(M,\omega) \xrightarrow{\mu} \Delta$, $f\downarrow$, $(M',\omega') \xrightarrow{\mu'}$

辛等变微分同胚

Remark: $\rho_1^*\omega_1=\omega_0$ $\rho_1^* f^*\omega'=\omega$ $(\boxed{f\circ\rho_1})^*\omega'=\omega$

proof: (分析)(Moser trick)

找 $\rho_t: M\to M$. $\rho_t^*\omega_t=\omega_0$

$\frac{d}{dt}\rho_t^*\omega_t=0$

$\omega_t=(1-t)\omega_0+t\omega_1$

$\frac{d}{dt}\omega_t=\omega_1-\omega_0$

$d\omega_t=(1-t)d\omega_0+t\,d\omega_1=0$

$(\frac{d}{dt}\rho_t^*)\omega_t+\rho_t^*(\frac{d}{dt}\omega_t)=0$

$\rho_t^* L_{V_t}\omega_t+\rho_t^*(\omega_1-\omega_0)=0$

$\rho_t^*(i_{V_t}d\omega_t+d\,i_{V_t}\omega_t)+\rho_t^*(\mu^*d\lambda)=0$

$\rho_t^* d(i_{V_t}\omega_t+\mu^*\lambda)=0$

claim 2. ω_t 非退化.

$i_{V_t}\omega_t+\mu^*\lambda=0$

$\Rightarrow$ $V_t=$? $V_t=\sum X_i\frac{\partial}{\partial x_i}+Y_i\frac{\partial}{\partial y_i}$

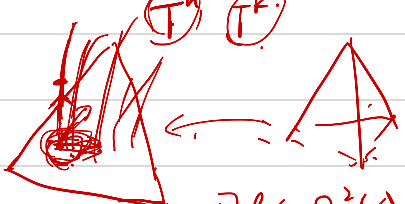
$\begin{cases}\frac{d\rho_t}{dt}=V_t\circ(\rho_t)\\ \rho_0=id\end{cases}$

$\Rightarrow \rho_t: M\to M$. ($M$ 紧)

claim: ① $\omega_1-\omega_0=\mu^*d\lambda$. $d\mu^*\lambda$

是 Δ 上 1-form

proposition:



T^n T^k

∃ $\beta\in\Omega^2(\Delta)$

$\omega_1-\omega_0=\mu^*\beta$.

$f^*\omega'$ ω

$\mu(M)=\mu'(M')$

$\omega_1=\omega_0+\mu^*\beta$ Δ 可缩

$\omega_1-\omega_0=\mu^*d\lambda$ $\lambda\in\Omega^1(\Delta)$

claim 2. $\omega_t = (1-t)\omega_0 + t\omega_1$ on μ 非退化:

proof: $T_xM = \ker T_x\mu^t \oplus W \xrightarrow{\simeq} \operatorname{Im} T_x\mu^t$

$= (T_xG\cdot x)^{\omega_t^\perp} \oplus W$

$= (T_xG\cdot x)^{\omega_t^\perp}/T_xG\cdot x \oplus T_xG\cdot x \oplus W$

$W \cong \operatorname{Im} T_x\mu^t$

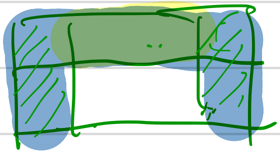
G - 交换 有效

$\Downarrow$

$(T_xG\cdot x) \subseteq (T_xG\cdot x)^{\omega_t^\perp}$

$\mathfrak{g}_x^0 = \{\xi\in\mathfrak{g}^* \mid \langle\xi,\eta\rangle = 0 \ \forall \eta\in\mathfrak{g}_x\}$

$\mathfrak{g}_x$ = G_x 的李代数.



$$\begin{array}{c|ccc}
 & T_x(G\cdot x)^{\omega_t^\perp}/T_xG\cdot x & T_xG\cdot x & W \\
\hline
T_x(G\cdot x)^{\omega_t^\perp}/T_xG\cdot x & (\omega_V) & 0 & * \\
T_xG\cdot x & 0 & 0 & Id \\
W & * & Id & *
\end{array}$$

$\omega(T_xG\cdot x, W)$

$= \langle \mathfrak{g}_x^0, \mathfrak{g}/\mathfrak{g}_x\rangle$

$= Id$

$T_xG\cdot x \cong \mathfrak{g}/\mathfrak{g}_x \qquad W \cong \mathfrak{g}_x^0$

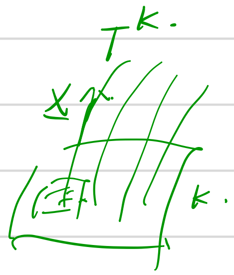
$\Downarrow$ 非退化

claim 3. $\mu\circ\rho_t = \mu$

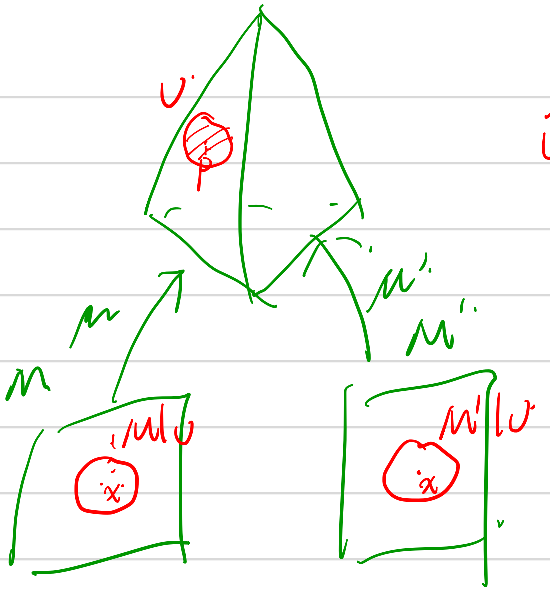
proof: 即证 $d\mu(v_t) = 0$

$$\begin{aligned}
i_{v_t} d\mu^X &= i_{v_t} i_X \omega_t \\
&= -i_X i_{v_t}\omega_t \\
&= i_X \mu^*\lambda \\
&= 0
\end{aligned}$$

$\lambda \in \Omega^1(\Delta)$



proof of ①, ② (孝变微分同胚)
Idea:



$p \in \tilde{U} \subseteq \mathbb{R}^n$ $p \in k$维面F

$\tilde{U} \cap \bar{F} = U$

proposition IV4.16.

F是Δ中k-维面 $(k \le n)$

则 $\mu|_{\mu^{-1}(\bar{F})} : \mu^{-1}(\bar{F}) \to \bar{F}$

是纤维为T^k的纤维化.

Remark: $\mu^{-1}(\bar{F}) \cong \bar{F} \times T^k$

$M|_U \cong \boxed{U \times T^k} \times \boxed{B^{2n-2k}(\varepsilon)}$

$M'|_U \cong U \times T^k \times B^{2n-2k}(\varepsilon)$

$(J_1, \cdots, J_k, e^{i\theta_1}, \cdots, e^{i\theta_k}, z_1, \cdots, z_{n-k})$

$\boxed{\omega} = \sum dJ_i \wedge d\theta_i + \sum dx_i \wedge dy_i$ 阿诺德

作用 $T^n = T^k \times T^{n-k} \times (\quad) \to (\quad)$. 刘维尔

$(e^{i\varphi_1}, \cdots, e^{i\varphi_k}, u_1, \cdots, u_{n-k})(J_1, \cdots, J_k, e^{i\theta_1}, \cdots, e^{i\theta_k}, z_1, \cdots, z_{n-k})$

$= (J_1, \cdots, J_k, e^{i(\theta_1+\varphi_1)}, \cdots, e^{i(\theta_k+\varphi_k)}, u_1 z_1, \cdots, u_{n-k} z_{n-k})$

ω: $i_X \omega = d\mu^X$

$\mu^X = (\;?\;, \;?\;, \frac{1}{2}|z_1|^2, \cdots, \frac{1}{2}|z_{n-k}|^2) + C.$

$T^k \times \quad U \times T^k.$

$X = (\frac{\partial}{\partial\varphi_1}, \cdots, \frac{\partial}{\partial\varphi_k})$ $\quad \underline{X} = (0, \cdots, 0, \frac{\partial}{\partial\theta_1}, \cdots, \frac{\partial}{\partial\theta_k})$

$\sum_{j=1}^{k} dJ_j \wedge d\theta_j((0, \cdots, 0, \frac{\partial}{\partial\theta_1}, \cdots, \frac{\partial}{\partial\theta_k}), Y) = d\tilde{\mu}^X(Y)$

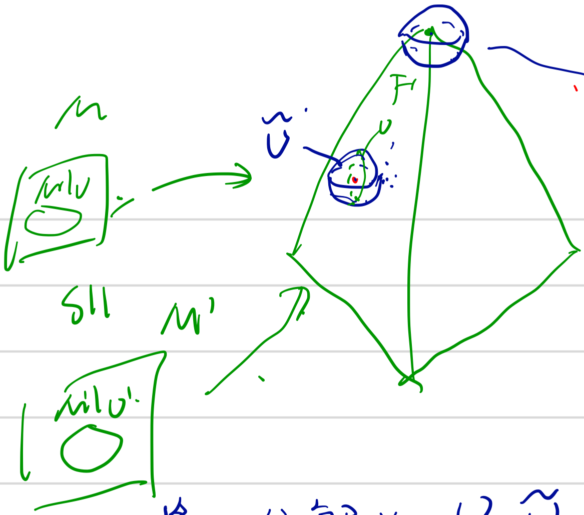
$Y = \frac{\partial}{\partial\theta_i}$, Left $= dJ_i(0)\, d\theta_i(\frac{\partial}{\partial\theta_i}) - d\theta_i(\frac{\partial}{\partial\theta_i})\, dJ_i(\frac{\partial}{\partial\theta_i}) = 0$

$Y = \frac{\partial}{\partial J_i}$, Left $= dJ_i(0)\, d\theta_i(\frac{\partial}{\partial J_i}) - d\theta_i(\frac{\partial}{\partial\theta_i})\, dJ_i(\frac{\partial}{\partial J_i}) = -1$

$\Rightarrow d\mu^X(\frac{\partial}{\partial\theta_i}) = 0$ $\quad \Rightarrow \tilde{\mu}^X = (0, \cdots, 0, -J_1, \cdots, -J_k)$

$d\mu^X(\frac{\partial}{\partial J_i}) = -1$

$\Rightarrow \mu^X = p + (-J_1, \cdots, -J_k, \frac{1}{2}|z_1|^2, \cdots, \frac{1}{2}|z_{n-k}|^2)$



$\Delta \subseteq \mathbb{R}^n$ $n=3$.

$T^{k} \times U \times B^{2n-2k}(\varepsilon)$

$T^{k+1} \times U \times B^{2n-2k-2}(\varepsilon)$

将Δ分解为 $\bigsqcup \tilde{U}_{i,F}$ U表示F所在的面维数

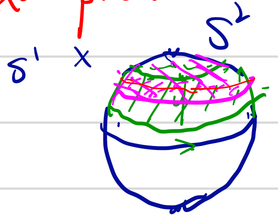
s.t. ① $\tilde{U}_{i,F} \cap \tilde{U}_{i,F'} = \emptyset$

② $\tilde{U}_{i,F} \cap F \ (= U_{i,F})$ 在F中相对闭，

③ $\mu^{-1}(\tilde{U}_{i,F})$ 中等变微分同胚可延拓到更高维面上

$\tilde{U}_{i,F}$, $\tilde{U}_{j,F}$.

Example:



$\xrightarrow{h}$ $\mathbb{R}^1$ $\tilde{U}$ 1-单形 $T^1 \times U \times \{0\}$ $n=1$ $k=1$

$\tilde{U} \cap \mathbb{R}^1 = U$

$\tilde{U}_0$ 是端点处的邻域 $T^0 \times \{point\} \times D^2$ $n=1$. $k=0$

$U = \tilde{U}_0 \cap \{point\} = \{point\}$

$U \times S^1 \to S^1 \times U$ in $\tilde{U}'$ 可延拓到 $\{point\} \times D^2$

Lemma: $\varphi: T^k \times U \times S^{2n-2k-1}(\varepsilon) \to T^k \times U \times S^{2n-2k-1}(\varepsilon)$

① φ 是 T^n 等变 diffeo

② $\mu \circ \varphi = \mu$.

那么 φ 可以延拓至 $\tilde{\varphi}: T^k \times U \times B^{2n-2k}(\varepsilon) \to T^k \times U \times B^{2n-2k}(\varepsilon)$

微分同胚. 满足 T^n 等变. $\mu \circ \tilde{\varphi} = \mu$.

proof: $\varphi(\underbrace{\theta_1, \dots, \theta_k}_{\theta}, \underbrace{J_1, \dots, J_k}_{J}, z_1, \dots, z_{n-k}) = (\theta + \theta_0, J, \psi_1, \dots, \psi_{n-k})$ $\quad \psi_i =$

$\mu \circ \varphi = \mu$

$\because \mu = p + (J, \tfrac{1}{2}|z_1|^2, \dots, \tfrac{1}{2}|z_{n-k}|^2)$

$\mu \circ \varphi = p + (J, \tfrac{1}{2}|\psi_1|^2, \dots, \tfrac{1}{2}|\psi_{n-k}|^2)$

$\Rightarrow |\psi_i|^2 = |z_i|^2$ ②

又 $\because \varphi \circ T^n = T^n \circ \varphi$ $\quad (\theta, J, z_1, \dots, z_{n-k})$

左 $\varphi(\theta+\theta_0', J, u_1 z_1, \dots, u_n z_n)$

$= (\theta + \theta_0 + \theta_0', J, \psi_1(\dots, u_1 z_1, \dots, u_n z_{n-k}), \dots, \psi_n(\dots, u_1 z_1, \dots, u_n z_{n-k}))$

右 $T^n \circ \varphi = T^n(\theta + \theta_0, J, \psi_1, \dots, \psi_{n-k})$

$= (\theta + \theta_0' + \theta_0, J, u_1 \psi_1, \dots, u_n \psi_{n-k})$

$\Rightarrow \psi_j(\theta, J, u_1 z_1, \dots, u_{n-k} z_{n-k}) = u_j \psi_j(\theta, J, z_1, \dots, z_{n-k})$ ①

取 z_j 都是实数 x_j.

② $\Rightarrow |\psi_j|^2 = |x_j|^2$

① $\Rightarrow \psi_1(\theta, J, -x_1, \dots, x_{n-k}) = -x_1 \psi(\theta, J, x_1, \dots, x_{n-k})$

再变形.

$\exists f_1: S^{2n-2k-1}(\varepsilon) \to S^1$. $\quad |2n-2k-1| = 1$. $\quad$ ~~$\psi_1(\theta, J, x_1, \dots, x_{n-k})$~~

$x_1 f_1(\theta, J, x_1, \dots, x_{n-k}) = \psi_1(\theta, J, x_1, \dots, x_{n-k})$

② $\Rightarrow |\psi_1|^2 = |x_1|^2$ $\quad \therefore |f_1|^2 = 1$

① $\Rightarrow$ $f_1(\theta, J, -x_1, \dots, x_{n-k}) = f(\theta, J, x_1, \dots, x_{n-k})$

f_1 关于 x_1 是偶函数

Whitney $\Rightarrow f(\theta, J, x_1, \dots, x_{n-k}) = g(\theta, J, x_1^2, \dots, x_{n-k}^2) \to S^1$

$\psi_1(\theta, J, z_1, \dots, z_k) = z_1 g(\theta, J, z_1^2, \dots, z_{n-k}^2)$

$= z_1 \exp(i h(\theta, J, z_1^2, \dots, z_{n-k}^2))$