Exercise IV.4:

Let $k \in \mathbb{N}$ be an integer, consider the subset of $\mathbb{P}^1 \times \mathbb{P}^2$:

$$W_k = \left\{ ([a,b],[x,y,z]) \in \mathbb{P}^1 \times \mathbb{P}^2 \mid a^k y = b^k x \right\}$$

1: Prove that $W_k$ is a submanifold of $\mathbb{P}^1 \times \mathbb{P}^2$, it coincides with definition of $W_k = (S^3 \times \mathbb{P}^1)/S^1$.

2: Endow $\mathbb{P}^1, \mathbb{P}^2$ with the standard Fubini-Study forms, then $W_k$ is also a symplectic manifold, Let $T^2$ acts on $\mathbb{P}^1 \times \mathbb{P}^2$ by

$$(u,v) \cdot ([a,b],[x,y,z]) = ([ua,b],[u^k x, y, vz])$$

Prove this action is a Hamiltonian action with the momentum mapping:

$$\mu([a,b],[x,y,z]) = \left( \frac{1}{2}\left( \frac{|a|^2}{|a|^2+|b|^2} + k \frac{|x|^2}{|x|^2+|y|^2} \right), \frac{1}{2} \frac{|z|^2}{|x|^2+|y|^2+|z|^2} \right)$$

3): Check that the action restricts to a $T^2$-action on $W_k$ and determine the image of the moment mapping.

Proof: 1): There're 3 covers on $\mathbb{P}^2$ & on $\mathbb{P}^1$, namely:

$$U_1 = \{ [a,b] \mid a \neq 0, b \in \mathbb{C} \} \qquad V_1 = \{ [x,y,z] \mid x \neq 0, (y,z) \in \mathbb{C}^2 \}$$

$$U_2 = \{ [a,b] \mid a \in \mathbb{C}, b \neq 0 \} \qquad V_2 = \{ [x,y,z] \mid y \neq 0, (x,z) \in \mathbb{C}^2 \}$$

$$V_3 = \{ [x,y,z] \mid z \neq 0, (x,y) \in \mathbb{C}^2 \}$$

hence there will be 6 covers on $W_k$, namely $W_k \cap (U_i \times V_j)$, however, for

$$W_k \cap (U_1 \times V_2) = \left\{ ([a,b],[x,y,z]) \mid (a,y) \in (\mathbb{C}^*)^2, (b,x,z) \in \mathbb{C}^3, a^k y = b^k x \right\}$$

is not empty if and only if $b \neq 0$ or $x \neq 0$, if $b \neq 0$, $W_k \cap (U_1 \times V_2) \subset W_k \cap (U_2 \times V_2)$, if $x \neq 0$, $W_k \cap (U_1 \times V_2) \subset W_k \cap (U_1 \times V_1)$, and same discussion for $W_k \cap (U_2 \times V_1)$, hence there are just 4 covers in $W_k$, namely:

$$O_1 = W_k \cap (U_1 \times V_1) \qquad O_3 = W_k \cap (U_1 \times V_3)$$

$$O_2 = W_k \cap (U_2 \times V_2) \qquad O_4 = W_k \cap (U_2 \times V_3)$$

On each $O_i$, one can define mapping:

$$\varphi_1 : O_1 \xrightarrow{\cong} \mathbb{C}^2 \qquad\qquad \varphi_2 : O_2 \xrightarrow{\cong} \mathbb{C}^2$$

$$\left([1,\tfrac{b}{a}],[1,\tfrac{b^k}{a^k},\tfrac{z}{x}]\right) \longmapsto \left(\tfrac{b}{a}, \tfrac{z}{x}\right) \qquad \left([\tfrac{a}{b},1],[\tfrac{a^k}{b^k},1,\tfrac{z}{y}]\right) \longmapsto \left(\tfrac{a}{b}, \tfrac{z}{y}\right)$$

$$\varphi_3 : O_3 \xrightarrow{\cong} \mathbb{C}^2 \qquad\qquad \varphi_4 : O_4 \xrightarrow{\cong} \mathbb{C}^2$$

$$\left([1,\tfrac{b}{a}],\left[\tfrac{x}{z},\left(\tfrac{b}{a}\right)^k\tfrac{x}{z},1\right]\right)\mapsto\left(\tfrac{b}{a},\tfrac{x}{z}\right) \qquad \left([\tfrac{a}{b},1],\left[\left(\tfrac{a}{b}\right)^k\tfrac{y}{z},\tfrac{y}{z},1\right]\right)\mapsto\left(\tfrac{a}{b},\tfrac{y}{z}\right)$$

these $(O_i, \varphi_i)$ gives a chart on $W_k$, hence it is a 2-dimensional complex manifold

To see it is equivalent with the quotient space $(S^3\times\mathbb{P}^1)/S^1$, we define a map $\psi : W_k \to (S^3\times\mathbb{P}^1)/S^1$ piecewisely:

$$\psi_1 : O_1 \longrightarrow (S^3\times\mathbb{P}^1)/S^1,$$

$$\left([1,\tfrac{b}{a}],\left[1,\left(\tfrac{b}{a}\right)^k,\tfrac{z}{x}\right]\right)\longmapsto\left[\frac{(a,b)}{\sqrt{|a|^2+|b|^2}},[x,z]\right]$$

$$\vdots$$

$$\psi_4 : O_4 \longrightarrow (S^3\times\mathbb{P}^1)/S^1$$

just to check these $\psi_i$ are diffeomorphism onto image, and they are gluable to a global map $\psi : W_k \xrightarrow{\cong} (S^3\times\mathbb{P}^1)/S^1$, i.e. $\psi_i|_{O_i\cap O_j} = \psi_j|_{O_i\cap O_j}$.

2): Note that this action is the descending of the action:

$$T^2\times\mathbb{C}^2\times\mathbb{C}^3 \longrightarrow \mathbb{C}^2\times\mathbb{C}^3$$

$$(u,v)\cdot\big((a,b),(x,y,z)\big) := \big((ua,b),(u^kx,y,vz)\big).$$

Let $X=(\alpha,\beta)\in \mathrm{Lie}(T^2)\cong\mathbb{R}^2$, hence the fundamental vector field associate to $X$ is:

$$\underline{X}\big((a,b),(x,y,z)\big) = \frac{d}{dt}\Big|_{t=0}(\exp tX)\big((a,b),(x,y,z)\big)$$

$$= \big((i\alpha a, 0),(i\alpha kx, 0, i\beta z)\big)$$

Hence the moment map $\tilde{\mu}_X$ on $(\mathbb{C}^2\setminus 0)\times(\mathbb{C}^3\setminus 0)$ is: ($X=(\alpha,\beta)\in \mathrm{Lie}(T^2)$)

$$\tilde{\mu}_X\big((a,b),(x,y,z)\big) = \frac{\alpha}{2}|a|^2+\frac{\alpha k}{2}|x|^2+\frac{\beta}{2}|z|^2,$$

which satisfies: $i_{\underline{X}}\omega = -(d\tilde{\mu}_X)$, hence the moment map on $\mathbb{P}^1\times\mathbb{P}^2$ is:

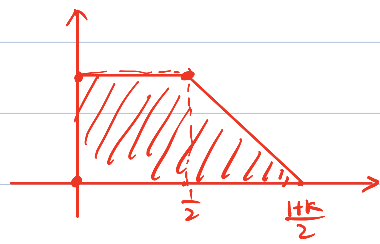
$$\mu([a,b],[x,y,z]) = \left(\frac{1}{2}\frac{|a|^2}{|a|^2-|b|^2}+\frac{k}{2}\frac{|x|^2}{|x|^2+|y|^2+|z|^2},\ \frac{1}{2}\frac{|z|^2}{|x|^2+|y|^2+|z|^2}\right)$$

3): The restriction $T^2$-action on $W_k$ is again Hamiltonian, with the moment mapping: $\mu' = \mu \circ i$, where $i: W_k \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ is the inclusion, to deduce the image of $\mu'$, it suffices to deduce the fixed points of $\mu$, and find out the fixed points which are in $W_k$, then compute its image.

the fixed points of $\mu$ are (which belongs $W_k$):

$$([a,0],[x,0,0]),\ ([a,0],[0,0,z]),\ ([0,b],[0,y,0]),\ ([0,b],[0,0,z]).$$
$$\downarrow \mu \qquad\qquad \downarrow \mu \qquad\qquad \downarrow \mu \qquad\qquad \downarrow \mu$$
$$\left(\tfrac{1+k}{2},0\right) \qquad \left(\tfrac{1}{2},\tfrac{1}{2}\right) \qquad (0,0) \qquad \left(0,\tfrac{1}{2}\right)$$

hence $\mu'$ has the image:




## Exercise 4.17.

Let $f(t) = \frac{1}{1+t^{-k}}$, $\quad U_1 = \{[u,1] \mid u \in \mathbb{C}\}$, $U_2 = \{[1,u] \mid u \in \mathbb{C}\}$ be the covers on $\mathbb{P}^1$.

1): Show that $\quad \varphi: \mathbb{C} \times \mathbb{P}^1 \longrightarrow W_k$

$$(u,[v,w]) \longmapsto \left([u,1],\ [f(|u|)u^k v,\ f(|u|)v,\ w]\right)$$
$$\psi: (u,[v,w]) \longmapsto \left([1,u],[f(|u|)v,\ f(|u|)u^k v,\ w]\right).$$

2): Show that the loop

$$\gamma: S^1 \longrightarrow \mathrm{Diff}(S^2)$$
$$z \longmapsto \left([v,w] \longmapsto [z^k v, w]\right)$$

is nontrivial if and only if $k$ is odd, and deduce that $W_k$ is either diffeomorphic to $S^2 \times S^2$ or $\tilde{\mathbb{P}}^2$, the $\mathbb{P}^2$ blown up at a point.

Proof: 1): Just to check that the $\varphi$, $\psi$ are injective, and onto $\pi^{-1}(U_1)$, $\pi^{-1}(U_2)$ respectively.

It defines a local trivilization on $W_k$. and

$$\psi^{-1} \circ \varphi\,(u,[v,w]) = \left(u,[u^k \cdot |u|^k v,\ w]\right)$$

is a transition function, hence

$$\gamma: S^1 \longrightarrow \mathrm{Diff}(S^2)$$
$$z \longmapsto \left([v,w] \longmapsto [z^k v, w]\right)$$

is the clutching function.

2): To study the triviality of the loop $\gamma$ in $\mathrm{Diff}(S^2)$, we use the fact that $\pi_1(\mathrm{Diff}(S^2)) \simeq \pi_1(SO(3))$.

Recall that, there is the homeomorphism from $\mathbb{P}^1 \longrightarrow S^2$.

$$[v,w] \longmapsto \left( \frac{\mathrm{Re}(v\bar{w})}{|v|^2+|w|^2}, \frac{\mathrm{Im}(v\bar{w})}{|v|^2+|w|^2}, \frac{|v|^2-|w|^2}{|v|^2+|w|^2} \right)$$

$$:= (x, y, z)$$

now, if we write $z = e^{i\theta}$, then, our $\gamma$ is:

$$\gamma(\theta) = \Big( (x,y,z) \longmapsto (\cos k\theta\, x - \sin k\theta\, y, \sin k\theta\, x + \cos k\theta\, y, z) \Big)$$

hence, write it in the matrix form is:

$$\gamma(\theta) = \begin{pmatrix} \cos k\theta & -\sin k\theta & 0 \\ \sin k\theta & \cos k\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \pi_1(SO(3))$$

to show the triviality of $\gamma(\theta)$ in $SO(3)$, we use the lifting curve $\tilde{\gamma}$ in $SU(2)$, note that, there is a 2-covering map $\varphi: SU(2) \longrightarrow SO(3)$, $\varphi(A) = (\varphi(A))_{ij}^{3\times 3}$, where:

$$\varphi(A)_{ij} = \frac{1}{2} \mathrm{tr}(\sigma_i A \sigma_j A^*)$$

where $\sigma_{1,2,3}$ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Computing the lifting curve $\tilde{\gamma}$ in $SU(2)$ is:

$$\tilde{\gamma}(\theta) = \begin{pmatrix} e^{-\frac{ik\theta}{2}} & 0 \\ 0 & e^{\frac{ik\theta}{2}} \end{pmatrix}$$

One can see that, if $k$ is even, $\tilde{\gamma}$ is contractible, hence $\gamma$ is trivial, non-trivial if $k$ is odd, hence $W_k$ are either diffeomorphic to $W_0 = S^2 \times S^2$, or $W_1 = \widetilde{\mathbb{P}}^2$, the $\mathbb{P}^2$ blown up at a point.