Convexity Theorem in Kähler case.

Assume we have a torus acting in a Hamiltonian way on a compact Kähler manifold W endowed with a Riemann metric and a complex structure.

We require the torus action also preserves Riemann metric. Hence the action extends to a complexificated torus action.

$$(\mathbb{C}^*)^n = \{(e^{u_1}t_1, \dots, e^{u_n}t_n) \mid (t_1 \dots t_n) \in T^n,\ u_i \in \mathbb{R}\}$$
$$\cong \mathbb{R}^n \times T^n.$$

Atiyah proved a refinement of his convexity theorem.

Thm. W: compact, connected Kähler manifold, endowed with a Hamiltonian action of torus T^n, with moment map μ. Let $V \subset W$ be an orbit of the complexified action of $(\mathbb{C}^*)^n$. Let $Z_1, \dots, Z_k$ be the components of the fixed points of T^n in W which intersect $\bar{V}$. Then $\mu(\bar{V})$ is the convex hull of $c_i = \mu(Z_i)$

Eg. Ex15. Let $\mathcal{D}$ be the mfd of complete flags in $\mathbb{C}^3$:

$$\mathcal{D} = \{0 \subset l \subset P \subset \mathbb{C}^3\}$$

Then $\mathcal{D} = \{([a,b,c],[x,y,z]) \in \mathbb{CP}^2 \times \mathbb{CP}^2 \mid ax+by+cz=0\}$

Endowed $\mathcal{D}$ with the induced Kähler structure

$T^2 = \{(t_1, t_2, t_3) \in T^3 \mid t_1 t_2 t_3 = 1\}$ acts on $\mathcal{D}$ by

$(t_1, t_2, t_3) \cdot ([a,b,c], [x,y,z])$

$$= ([t_1 a, t_2 b, t_3 c], [t_1^{-1} x, t_2^{-1} y, t_3^{-1} z])$$

This is a Hamiltonian action with moment map

$$\mu([a,b,c],[x,y,z]) = \frac{1}{2}\left(\frac{|a|^2}{|a|^2+|b|^2+|c|^2} - \frac{|x|^2}{|x|^2+|y|^2+|z|^2}, \right.$$

$$\frac{|b|^2}{|a|^2+|b|^2+|c|^2} - \frac{|y|^2}{|x|^2+|y|^2+|z|^2},$$

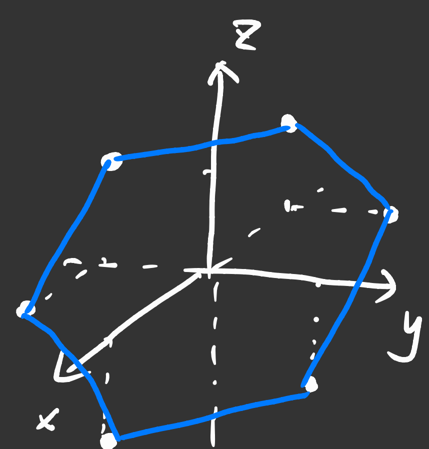
$$\left. \frac{|c|^2}{|a|^2+|b|^2+|c|^2} - \frac{|z|^2}{|x|^2+|y|^2+|z|^2} \right)$$

The image of μ:

Fixed point $\begin{cases} [t_1 a, t_2 b, t_3 c] = [a,b,c] \\ [t_1^{-1} x, t_2^{-1} y, t_3^{-1} z] = [x,y,z] \\ ax+by+cz=0 \end{cases}$ $\forall t = (t_1, t_2, t_3) \in T^2$.

Fixed point =

$$\begin{array}{lcl} ([1,0,0], [0,1,0]) & & \frac{1}{2}(1,-1,0) \\ ([1,0,0], [0,0,1]) & \xrightarrow{\mu} & \frac{1}{2}(1,0,-1) \\ ([0,1,0], [0,1,0]) & & \frac{1}{2}(-1,1,0) \\ ([0,1,0], [0,0,1]) & & \frac{1}{2}(0,1,-1) \\ ([0,0,1], [0,1,0]) & & \frac{1}{2}(-1,0,1) \\ ([0,0,1], [0,0,1]) & & \frac{1}{2}(0,-1,1) \end{array}$$



Complexified action of the torus

$$(\mathbb{C}^*)^2 = \{ (z_1, z_2, z_3) \in (\mathbb{C}^*)^3 \mid z_1 z_2 z_3 = 1 \}$$

$(z_1, z_2, z_3) \cdot ([a,b,c],[x,y,z])$

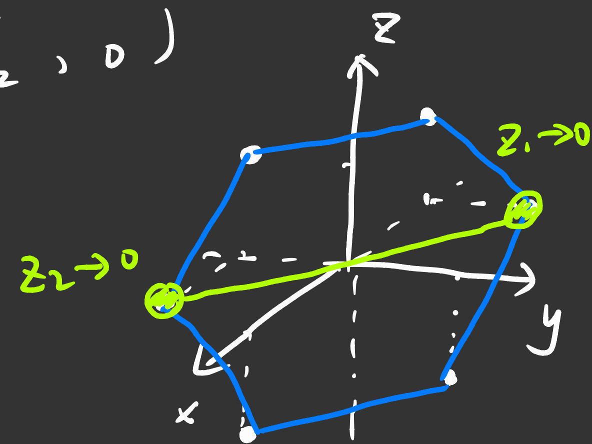
$$= ([z_1 a, z_2 b, z_3 c], [z_1^{-1}x, z_1^{-1}y, z_1^{-1}z])$$

Consider the image of the orbit of $([1,1,0],[1,-1,0])$

Orbit : $\{([z_1, z_2, 0],[z_2, -z_1, 0]) \mid z_1, z_2 \in \mathbb{C}^*\}$

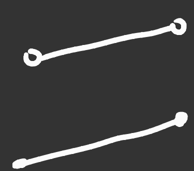
$$\mu = \frac{1}{2}\left(\frac{|z_1|^2 - |z_2|^2}{|z_1|^2+|z_2|^2}, \frac{|z_2|^2 - |z_1|^2}{|z_1|^2+|z_2|^2}, 0\right)$$

$z_2 \to 0$: $([1,0,0],[0,1,0])$



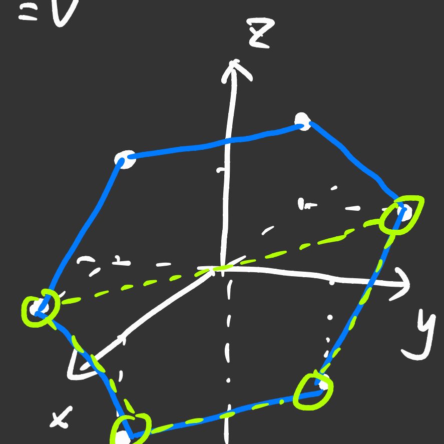
$V \to$

$\bar{V} \to$



Consider the image of the orbit of $([1,1,0],[1,-1,1])$

Orbit : $\{([z_1, z_2, 0],[z_1^{-1}, -z_2^{-1}, z_1 z_2])\} = V$



Pf of Kähler case :

① $\mu(\bar{V}) \subset \widehat{\{c_1, \dots c_k\}}$. $c_i = \mu(z_i)$. $\mu = (f_1, \dots f_n)$

Recall $f = \sum \lambda_i f_i$, for generic choice of λ, f generates T^n-action. So critical points of f is exactly common zero of df_i. So $\sum \lambda_i f_i$ reaches its maximum at $\mu(z_i)$, which imply $Im(\mu) \subset \widehat{\{c_1, \dots c_k\}}$.

φ^t : flow of ∇f.

Similarly, we consider $f = \sum \lambda_i f_i$. $\forall x \in V$, we have

Lemma. Let x_∞ be a limit point of $\varphi^t(x)$ as $t \to \infty$. Then

(i) x_∞ lies on a critical submanifold N of f.

(ii) $\lim_{t \to +\infty} f(\varphi^t(x))$ exists and is a constant $f(x_\infty)$ independent of x.

(iii) $f(x_\infty) = \sup_{x \in V} f(x)$

Pf. (i)

(ii) $f(\varphi^t(x)) : \mathbb{R} \to \mathbb{R}$ is bounded increasing function hence it has a limit as $t \to \infty$, which is equal to $f(x_\infty)$.

Now Let Z be the component of critical points of f to which x_∞ belongs. $\Rightarrow f|_Z = \text{const}$.

Moreover, Z is an invariant submanifold for the $(\mathbb{C}^*)^n$-action.

$$\forall z \in (\mathbb{C}^*)^n, \quad \lim_{t \to \infty} f(\varphi^t(z \cdot x)) = f(z \cdot x_\infty) = f(x_\infty)$$

So that, V being an orbit, $f(x_\infty)$ doesn't depend on anything and we have $f(x_\infty) = \sup_{x \in V} f(x)$.

Back to the proof, we choose λ_i's so that the function f generates the whole T^n-action.

Then the Z above is one of the Z_j's

Thus $\quad \sup_{x \in V} f(x) = \sup_{j = 1, \dots, k} f(Z_j)$

$$\Rightarrow \mu(\bar{V}) \subset \widehat{\{c_1, \dots, c_k\}} = P.$$

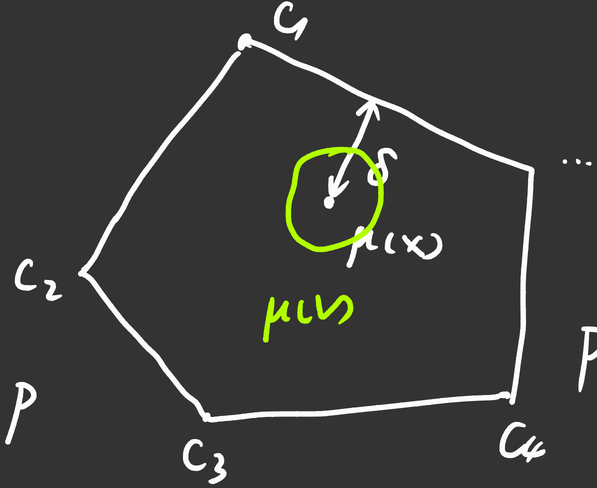
Reverse inclusion:

we'll prove : $\forall x \in V$, let $\delta > 0$ be the distance of $\mu(x)$ to ∂P

$\mu(V)$ contains the $B(\mu(x), \frac{1}{2}\delta)$.

So $\mu(V)$ is the whole interior of P

$$\mu(\bar{V}) = P$$

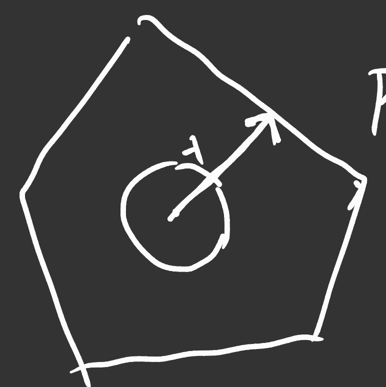


By adding the constant vector, we can assume $\mu(x) = $ origin of $\mathbb{R}^n$.

Consider the support function of P

$$\sigma : S^{n-1} \longrightarrow \mathbb{R}$$

$$\lambda = (\lambda_1, \dots \lambda_n) \mapsto \sup_{\xi \in P} \left(\sum \lambda_i \xi_i \right)$$



Now $\delta = \inf\limits_{\lambda \in S^{n-1}} \sigma(\lambda)$

Consider $f_\lambda = \sum \lambda_i f_i$, so

$$\sup_{x \in V} f_\lambda(x) = \sup_{j=1,\dots k} f_\lambda(z_j) = \sup_{j=1,\dots k} \underbrace{\sum \lambda_i f_i(z_j)}_{\lambda \cdot f(z_j)} = \sigma(\lambda)$$

We look now at the trajectory of the gradient flow φ_λ^t of f_λ through x.

$t=0$ $\quad \varphi_\lambda^0(x) = x \quad f_\lambda(\varphi_\lambda^0(x)) = f_\lambda(x) = \sum \lambda_i \underline{f_i(x)} = 0$

$t \to +\infty \quad f_\lambda(\varphi_\lambda^t(x)) \to \sigma(\lambda)$

$\delta = \inf \sigma(\lambda)$

$\exists\, t_\lambda \in [0, +\infty),\ f(\varphi_\lambda^t(x)) = \delta/2.$

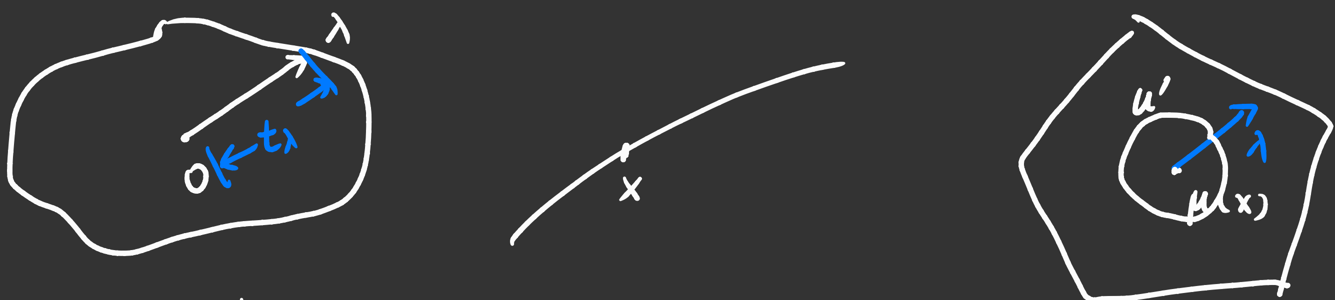
This gives us a $S^{n-1} \to \mathbb{R}$

$\lambda \mapsto t_\lambda$

Construct a nbhd of 0 in $\mathbb{R}^n$:

$\{ r\lambda \mid \lambda \in S^{n-1},\ r \le t(\lambda) \} =: \mathcal{U}.$

$$\mathcal{U} \subset \mathbb{R}^n \xrightarrow{\exp} V \xrightarrow{\mu} \mathcal{U}' \subset \mathbb{R}^n$$



$$y \longmapsto \exp(y)\cdot x \longmapsto \mu(\exp y \cdot x)$$

$\forall\, \xi \in \partial\mathcal{U}',\ \exists \lambda,\ \|\lambda\| = 1,\ \underline{\sum \lambda_i \xi_i = \frac{\delta}{2}} = \lambda \cdot \xi$

$1 \quad \|\xi\| > \frac{\delta}{2}.$

(ii) For each open face σ of P, $\mu^{-1}(\sigma)$ in $\bar{V}$ consists of a single $(\mathbb{C}^*)^n$-orbit.

(iii) μ induces a homeomorphism of $\bar{V}/T$ onto P.

$$\mathrm{Lie}((\mathbb{C}^*)^n) = \mathbb{R}^n \times t^n.$$

$\|$

$\underline{\mathbb{R}^n \times T^n}$

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Application: a theorem of Kushnirenko on monominal equations.

Consider a finite set $S \subset \mathbb{Z}^n$ of multi-exponents and the system of n equations and n unknowns.

$$\sum_{a\in S} C_a^j z^a = 0, \quad 1\le j\le n, \quad \underset{a\in S}{C_a^j \in \mathbb{C}}, \quad z\in(\mathbb{C}^*)^n.$$

The number of solutions for a generic enough choice of C_a^j

$$N(S) = n!\ \mathrm{Vol}(\hat{S})$$

Eg. ① $n=1$. $S=\{k_1,\dots k_n \mid \underline{k_1}\le k_2\le\cdots\le k_n\} \underset{\text{finite}}{\subset} \mathbb{Z}$.

$$\sum_{i=1}^n C_{k_i} z^{k_i} = 0 \Rightarrow z^{k_1}\left(\sum C_{k_i} z^{k_i-k_1}\right)=0$$

It has k_n-k_1 solutions (non-zero)

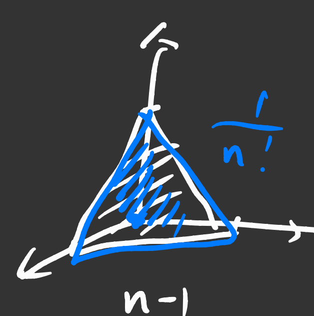
$N(S) = 1!\ \mathrm{vol}(\hat{S}) = k_n - k_1$

② $\forall n\in\mathbb{N}$. $S=\{e_1,\dots e_n\}$, $e_i=(0,\dots \underset{i\text{th}}{1}, 0\cdots 0)$

$$\Rightarrow \sum_{i=1}^n C_i^j z^{e_i} = 0 \quad .\ 1\le j\le n$$

$$z=(z_1,\dots z_n) \quad z^{e_i} = z_1^0\cdots \underline{z_i^1}\cdots z_n^0 = z_i$$

$$\Rightarrow \sum C_i^j z_i = 0 \qquad C\cdot z = 0$$



It has no non-zero solution.

$N(S) = n!\cdot \mathrm{vol}(\hat{S}) = 0.$

③ $\forall n,\ S = \{e_1, \dots, e_n, 0\}$ $\quad \sum C_i^j z_i + \underline{C_0^j} = 0$

$z^0 = z_1^0 \cdots z_n^0 = 1.$ It has unique non-zero solution

$N(S) = n!\, \mathrm{vol}(\hat{S}) = n! \cdot \frac{1}{n!} = 1.$

pf. $N = \# S$ and $a_1, \dots, a_N$ be the elements of S.

$(\mathbb{C}^*)^n$ acts on $\mathbb{C}^N$ (or $\mathbb{CP}^{N-1}$) by

$$t \cdot (z_1, \dots, z_n) = (t^{a_1} z_1, \dots, t^{a_N} z_N)$$

$$t \cdot [z_1, \dots, z_n] = [t^{a_1} z_1, \dots, t^{a_N} z_N]$$

Consider the orbit of $[1, 1, \dots, 1] = X_S \subset \mathbb{CP}^{N-1}$

It is parametrised by $z_1 = t^{a_1}, \dots, z_N = t^{a_N}$. $t \in (\mathbb{C}^*)^n$

Its closure $\overline{X_S}$ is a complex algebraic (in particular symplectic) submanifold (Toric variety)

$$\dim(\overline{X_S} \setminus X_S) < n.$$

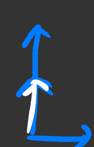
Let V be the projective subspace with equations

$$\underline{\sum C_a^j z_a = 0} \quad . \quad 1 \le j \le n \qquad \text{codim } n.$$

$$\left\{ \begin{array}{l} x - \overset{z_1}{y^2} = 0 \\ x = 1 \end{array} \right. \quad \begin{array}{l} \longrightarrow\ x - y^2 + 0 = 0 \longrightarrow z^{(1,0)} - z^{(0,2)} + 0\, z^{(0,0)} = 0 \\ \longrightarrow\ x + 0y^2 - 1 = 0 \longrightarrow z^{(1,0)} + 0 \cdots - z^{(0,0)} = 0 \end{array}$$

$(1,1)\ (1,-1)$

$S = \{(1,0), (0,2), (0,0)\}$



$\mathbb{Z}^2$

X_S (orbit) : $(x, y) \cdot [1,1,1] = [x, y^2, 1]$

V_S : $z_0 - z_1 = 0, \quad z_0 - z_2 = 0$

Solution

$$\begin{cases} x - y^2 = 0 \\ x = 1 \end{cases}$$

$X_S \cap V$

$[x, y^2, 1] \cap \{z_0 - z_1 = 0, z_0 - z_2 = 0\}$

projective subspace

in $\mathbb{C}P^N$, Intersection of X_S and V.

(i) X_S should be faithful. $(\mathbb{C}^*)^n \cdot [1, \cdots 1] = X_S$

claim : $\{a_i - a_j \mid a_i, a_j \in S\}$ generates the lattice $\mathbb{Z}^n$, then the orbit X_S is faithful.

faithful $\forall z, \tilde{z} \in (\mathbb{C}^*)^n$

$[z^{a_1}, \cdots z^{a_N}] = [\tilde{z}^{a_1}, \cdots, \tilde{z}^{a_N}] \Rightarrow z = \tilde{z}$.

$\forall e_k = (0, \cdots 0, \underset{k\text{-th}}{1}, 0 \cdots 0) \in \mathbb{Z}^n$, $\exists c_{ij}^k \in \mathbb{Z}$.

$$\sum_{i,j} c_{ij}^k (a_i - a_j) = e_k$$

$$\begin{cases} z^{a_1} = \lambda \tilde{z}^{a_1} \\ \vdots \\ z^{a_N} = \lambda \tilde{z}^{a_N} \end{cases}$$

for $\lambda \in \mathbb{C}^*$

$$z^{\sum c_{ij}^k (a_i - a_j)} = z^{e_k} = z_k$$

$$||$$

$$\tilde{z}^{\sum c_{ij}^k (a_i - a_j)} = \tilde{z}^{e_k} = \tilde{z}_k$$

Intersection.

codim $V = n$. we assume the coefficients to be generic, so that $V \pitchfork X_s$.

$$N(s) = I(V, X_s)$$

Thm [G&H : Principles of Algebraic Geometry]

[梅加强：流形与几何初步]

$$I(V, X_s) = \int_{X_s} PD[V].$$

Poincaré duality : M : compact. oriented n-mfd. the intersection pairing

$$H_k(M, \mathbb{Z}) \times H_{n-k}(M, \mathbb{Z}) \to \mathbb{Z}$$

is perfect / unimodular.

$$H_k(M, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_{n-k}(M, \mathbb{Z}), \mathbb{Z})$$

cell-decomposition of $\mathbb{C}P^N$:

$$\mathbb{C}P^N = (\mathbb{C}P^N \setminus \mathbb{C}P^{N-1}) \cup (\mathbb{C}P^{N-1} \setminus \mathbb{C}P^{N-2}) \cup \cdots \cup \mathbb{C}P^1 \setminus \mathbb{C}P^0 \cup \mathbb{C}P^0$$

V: generator of $H_{2(N-1-n)}(\mathbb{C}P^{N-1}, \mathbb{Z})$.

Its poincaré duality should be generator of $H^{2n}(\mathbb{C}P^{N-1}, \mathbb{Z})$

deRham : $H^k_{dR}(M, \mathbb{R}) \to \underline{H^k(M, \mathbb{R})}$

integral cohomology

$\underline{\alpha}$ k-form $\int_\sigma \alpha \in \mathbb{Z}$

Generator of $H^2_{dR}(\mathbb{C}P^{N-1}, \mathbb{Z})$ $\quad \mathbb{C}P^1 \subset \mathbb{C}P^{N-1}$.

$\int_{\mathbb{C}P^1} \omega_{FS} = \pi.$ $\quad \int_{\mathbb{C}P^1} \underset{\sim\sim}{\frac{\omega_{FS}}{\pi}} = 1 \in \mathbb{Z}$ $\quad \omega = \frac{\omega_{FS}}{\pi}$

$$H^{2n}_{dR}(\mathbb{C}P^{N-1}, \mathbb{Z}) = \omega^n.$$

V : $PD[V] = [\omega^n]$

$$N(S) = I(V, X_S) = \int_{X_S} PD[V] = \int_{X_S} \underline{[\omega^n]} = n! \, \mathrm{Vol}(X_S).$$

From Rmk IV 4.9. $\mu : X_S \to \mathring{P}$, $\mu^{-1}(\mathring{P}) \approx \mathring{P} \times T^n$.

and the component of moment map are already action coordinate.

From Rmk IV 3.9 $\quad \mathrm{Vol}(X_S) = m(\mathring{P}) = \mathrm{Vol}(\mathring{P}) = \mathrm{Vol}(P)$.

(Lebesgue measure)

$(\mathbb{C}^*)^n$ on $\mathbb{C}P^{N-1}$:

$$(t_1, \dots t_n) \cdot [z_1, \dots z_N] = [t^{a_1} z_1, \dots t^{a_N} z_N]$$

T^n on $\mathbb{C}P^{N-1}$ by

$$(t_1, \dots t_n) \cdot [z_1, \dots z_N] = [t^{a_1} z_1, \dots t^{a_N} z_N]$$

μ:

$$X_S \longrightarrow \mathbb{C}P^{N-1} \xrightarrow{\mu_0} (\mathbb{R}^*)^N \xrightarrow{proj} (\mathbb{R}^*)^n.$$

$$T^N \times \mathbb{C}P^{N-1} \longrightarrow \mathbb{C}P^{N-1}$$

$$((t_1, \dots, t_N), [z_1, \dots z_N]) \mapsto [t_1 z_1, \dots, t_N z_N)$$

$$\mu_0(z_1,\dots,z_n) = \frac{1}{|z_1|^2+\dots+|z_n|^2}\left(|z_1|^2,\dots,|z_n|^2\right)$$

$$T^n \hookrightarrow T^N$$

$$(t_1,\dots,t_n) \mapsto \left(t_1^{a_{11}} t_2^{a_{12}}\cdots t_n^{a_{1n}},\ \dots,\ t_1^{a_{N1}}\cdot t_2^{a_{N2}}\cdots t_n^{a_{Nn}}\right)$$

$\rightsquigarrow \ \mathfrak{t}^n \to \mathfrak{t}^N$

$$\mathbb{R}^n \ni (1,0,\dots,0) \mapsto (a_{11}, a_{21},\dots, a_{N1})$$

$$\vdots$$

$$(0,\dots,\underset{k}{1},\dots,0) \mapsto (a_{1k}, a_{2k},\dots, a_{Nk})$$

$\rightsquigarrow \ (\mathbb{R}^*)^N \to (\mathbb{R}^*)^n$

$$(1,0,\dots,0) \in \mathbb{R}^N \mapsto (a_{11}, a_{12},\dots, a_{1n}) = a_1$$

$$e_k \mapsto a_k$$

$$N(S) = I(X_S, V) = n!\,\mathrm{Vol}(X_S) = n!\,\mathrm{Vol}(\overset{\{a_1,\dots,a_n\}}{P}) = n!\,\mathrm{Vol}(\hat{S})$$

$$\Rightarrow N(S) = n!\,\mathrm{Vol}(\hat{S})$$

When $\{a_i - a_j \mid a_i, a_j \in S\}$ doesn't generate $\underline{\mathbb{Z}^n}$ }

If $\mathrm{rk}\ \mathrm{span}_{\mathbb{Z}}\{a_i - a_j\} < n$

If $\mathrm{rk}\ \mathrm{span}_{\mathbb{Z}}\underset{\overset{\shortparallel}{\Lambda}}{\{a_i - a_j\}} = n$. $\quad \mathbb{Z}^n$.

$$\mathbb{R}^n/\Lambda$$

~~B~~ $\overline{X_S} \to \hat{S}$