CLASSIFICATIONS OF COMPACT CONNECTED SYMPLECTIC SU(2)-4-MANIFOLDS

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ABSTRACT

Let

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \middle| \, |x|^2 + |y|^2 = 1 \right\} \cong \mathbb{S}^3$$

be the Lie subgroup of $\mathrm{GL}_2(\mathbb{C})$, with the Lie algebra:

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix} \middle| \, (a,b,c) \in \mathbb{R}^3 \right\} \cong \mathbb{R}^3$$

this can be identified with its dual Lie algebra $\mathfrak{su}(2)^*$ via

$$\mathfrak{su}(2) \xrightarrow{\cong} \mathfrak{su}(2)^*$$
$$A \mapsto \left(X \mapsto \frac{i}{2}\mathrm{tr}(A^* X) \right)$$

In this lecture, we will give the classifications of all connected compact symplectic 4-manifolds endowed with an effective $\mathrm{SU}(2)$-Hamiltonian action, the main theorem is as follows:

Theorem 1. *If M is a connected compact symplectic 4-manifolds endowed with an effective $\mathrm{SU}(2)$-Hamiltonian action, then M is either $\mathbb{P}^2$ or the Hirzebruch surface W_m with some m odd.*

1 Examples

1.1 SU(2)-action on $\mathbb{P}^2$

We define the $\mathrm{SU}(2)$ action on the complex projective space $\mathbb{P}^2$ by

$$A.[x,y,z] := \left[(x,y,z) \begin{pmatrix} A & \\ & 1 \end{pmatrix} \right]$$

This the restriction action of the standard $\mathrm{GL}_3(\mathbb{C})$ action on $\mathbb{P}^2$, hence is Hamiltonian, with $[0,0,1]$ as the fixed point.

The moment mapping can be computed as follows:
consider the inclusion

$$i : \mathrm{SU}(2) \hookrightarrow \mathrm{GL}_3(\mathbb{C})$$
$$A \mapsto \begin{pmatrix} A & \\ & 1 \end{pmatrix}$$

consider its differential mapping at I:

$$(di)_I : \mathfrak{su}(2) \longrightarrow \mathfrak{gl}_3(\mathbb{C})$$
$$X \mapsto \begin{pmatrix} X & \\ & 0 \end{pmatrix}$$

then try to compute the moment mapping $\mu' : \mathbb{C}^3 \longrightarrow \mathfrak{gl}_3(\mathbb{C})^*$ associated with the standard $\mathrm{GL}_3(\mathbb{C})$ action on $\mathbb{C}^3$, which is

$$\mu'(v) = \left(X \mapsto \frac{i}{2 \parallel v \parallel^2} v^* X v \right)$$

Then by taking the composition and descending it to the $\mathbb{P}^2$ level :

$$\begin{array}{ccccc} \mathbb{C}^3 & \xrightarrow{\mu'} & \mathfrak{gl}_3(\mathbb{C}) & \xrightarrow{(di)_I^*} & \mathfrak{su}(2)^* \\ \downarrow \pi & & \overset{\mu}{\nearrow} & & \\ \mathbb{P}^2 & & & & \end{array}$$

and that μ is by my computation:

$$\mu([v]) = \left(Y \mapsto \frac{i}{2 \parallel v \parallel^2} v^* \begin{pmatrix} Y & \\ & 0 \end{pmatrix} v \right)$$

1.2 SU(2)-action on Hirzebruch surfaces

Let $\mathbb{S}^3 = \{(a,b) \in \mathbb{C}^2 : |a|^2 + |b|^2 = 1\} \subset \mathbb{C}^2$ be the 3-dimensional sphere, consider the circle group action:

$$\mathbb{S}^1 \times \left(\mathbb{S}^3 \times \mathbb{C}\right) \longrightarrow \mathbb{S}^3 \times \mathbb{C}$$

$$u.((a,b),z) := ((ua,ub),u^{-m}z)$$

for some $m \in \mathbb{Z}$, consider the quotient space $\left(\mathbb{S}^3 \times \mathbb{C}\right)/\mathbb{S}^1$, it is a complex line bundle over $\mathbb{P}^1 \cong \mathbb{S}^3/\mathbb{S}^1$ via the projection:

$$\pi : \left(\mathbb{S}^3 \times \mathbb{C}\right)/\mathbb{S}^1 \longrightarrow \mathbb{P}^1$$

$$[(a,b),z] \mapsto [a,b]$$

which by our construction, is the bundle $\mathcal{O}(-m)$, it can be embedded into a trivial bundle:

$$i : \left(\mathbb{S}^3 \times \mathbb{C}\right)/\mathbb{S}^1 \hookrightarrow \mathbb{P}^1 \times \left(\mathbb{C}^2\right)^{\otimes m}$$

$$[(a,b),z] \mapsto ([a,b], z((a,b) \otimes ... \otimes (a,b)))$$

Now the SU(2) action on $\mathcal{O}(-m)$, choose $v \in \mathbb{S}^3 \subset \mathbb{C}^2$ and $z \in \mathbb{C}$:

$$A.[v,z] := [Av,z]$$

this action is precisely the restriction of SU(2) action on trivial bundle $\mathbb{P}^1 \times (\mathbb{C}^2)^{\otimes m}$:

$$A.([\ell],w) := \left([A\ell], A^{\otimes m}w\right)$$

for some $\ell \in \mathbb{C}^2 \setminus \{0\}$, and $w \in \mathbb{C}^{2^m}$, this is a symplectic action since it preserves the symplectic forms on each component.

Now, we wish to compactify $\mathcal{O}(-m)$ into a compact manifold, the way is to glue to it a copy of $\mathcal{O}(m)$:

$$W_m = (\mathcal{O}(-m) \sqcup \mathcal{O}(m)) \,/\, \left([(a,b),z] \sim [(a,b),z^{-1}]\right)$$

This is a sphere bundle over $\mathbb{P}^1$, the fibre is also $\mathbb{P}^1$, the result of this compactification is the quotient space $\left(\mathbb{S}^3 \times \mathbb{P}^1\right)/\mathbb{S}^1$, where the circle group action is given by:

$$\mathbb{S}^1 \times \left(\mathbb{S}^3 \times \mathbb{P}^1\right) \longrightarrow \mathbb{S}^3 \times \mathbb{P}^1$$

$$u.((a,b),[x,y]) := ((ua,ub),[x,u^m y])$$

This kind of surfaces is called the **Hirzebruch surface** W_m, the SU(2) action is defined by

$$A.[v,[\ell]] := [Av,[\ell]]$$

for some $v \in \mathbb{S}^3 \subset \mathbb{C}^2$ and $\ell \in \mathbb{C}^2 \setminus \{0\}$. We note from our inclusion that embeds $\mathcal{O}(-m)$ in to a trivial bundle, which will induce an inclusion:

$$j : \left(\mathbb{S}^3 \times \mathbb{P}^1\right)/\mathbb{S}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^{2^m}$$

$$[v,[x,y]] \mapsto \left([v], [x.v^{\otimes m}, y]\right)$$

Like before, the SU(2) action on W_m can be viewed as the restriction action on $\mathbb{P}^1 \times \mathbb{P}^{2^m}$[1], which is defined by:

$$A.([\ell],[w,z]) := \left([A\ell],[A^{\otimes m}w,z]\right)$$

This is a Hamiltonian action, and the moment map can be computed as follows:

Start form SU(2) action on the Euclidean space $\mathbb{C}^2 \times \mathbb{C}^{2^m+1}$, this action is composed from the action on each components,if denoted by μ_1', μ_2' and μ' for the moment maps associated with $\mathbb{C}^2$,$\mathbb{C}^{2^m+1}$ and their product respectively, then we have

$$\mu' = \mu_1' + \mu_2'$$

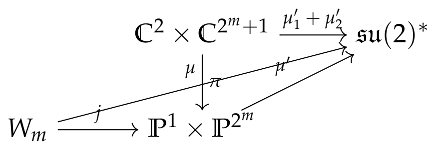
this follows from the matrix of symplectic form on the product space is the juxtapose of each component, which by my computation

$$\mu_1'(v) = \left(X \mapsto \frac{i}{2 \parallel v \parallel^2} v^* X v\right)$$

[1]Here, we will take $\mathbb{P}^{2^m}$ as the projectivization of $\left(\mathbb{C}^2\right)^{\otimes m} \oplus \mathbb{C}$

$$\mu_2'(w,z) = \left(X \mapsto \frac{i}{2(\| w \|^2 + |z|^2)} (\bar{w}, \bar{z}) \begin{pmatrix} X & & & \\ & \cdots & & \\ & & X & \\ & & & 1 \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \right)$$

then descending μ' to $\mathbb{P}^1 \times \mathbb{P}^{2^m}$ and by composition:



2 The Classification

Koszul's slice theorem is the only tool which can help us to do the classification under the Lie group action, the way is to consider the quotient map $f : M \longrightarrow M/G$, each point in the quotient M/G corresponds to an orbit, however, the principal orbits will take up the most room, so we just need to find which points are corresponding to the singular orbits or the exceptional orbits, a primitive image $f^{-1}(U)$ of U which contained $x \in M/G$ will be tubular neighbourhood of the corresponding orbit, then we analyse its stablizer G_x, by the Koszul's slice theorem, it will have the homotopy type $(G \times V_x)/G_x$, where V_x is the normal fibre, and the manifold M is the connected sum of some of these primitive images.

2.1 The principal orbits

By the proposition 3.2.10, the stablizer of a principal orbit is both discrete and closed subgroup of the maximal torus of $\mathrm{SU}(2)$, which is

$$\mathbb{S}^1 := \left\{ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \middle| \, |z| = 1 \right\}$$

hence the stablizer is a cyclic group $\mathbb{Z}_m$, and the principal orbits have dimension 3.

2.2 Exceptional orbits

The exceptional orbits have the same dimension as the principal orbits, thus it is also in dimension 3, to deduce its stablizer, we will need the equivariance of the moment map:

Lemma 1. *The moment map is equivariant under the G action on M and the coadjoint action, i.e the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{\mu} & \mathfrak{g}^* \\ \downarrow g & & \downarrow \mathrm{Ad}_g^* \\ M & \xrightarrow{\mu} & \mathfrak{g}^* \end{array}$$

Outlines of the proof: Just to check the commutativity of the infinitesimal case:

$$\begin{array}{ccc} M & \xrightarrow{\mu} & \mathfrak{g}^* \\ \downarrow \underline{X} & & \downarrow \mathrm{ad}_X^* \\ TM & \xrightarrow{d\mu} & \mathfrak{g}^* \end{array}$$

Now, for an exceptional orbit $\mathrm{SU}(2).x$ its image under the moment map is an coadjoint orbit in $\mathfrak{su}(2) \cong \mathbb{R}^3$, which is a symplectic leaf, hence has dimension 2, thus it is $\mathbb{S}^2$[2], hence the stablizer $\mathrm{SU}(2)_x$ is a subgroup of $\mathrm{SU}(2)_{\mu(x)}$, the later has dimension 1, hence it is $\mathbb{Z}_q$, a tubular neighbourhood has the homotopy type $(\mathrm{SU}(2) \times \mathbb{R})/\mathbb{Z}_q$, $\mathbb{Z}_q$ acts on $\mathbb{R}$ either by trivial action or the reflection, however, in the non-trivial case, that

[2] One need to check that the coadjoint orbit of SU(2) in dimension 2 is a sphere $\mathbb{S}^2$

yields a non-orientable normal bundle, hence its impossible[3], and all orbits in this neighbourhood have stablizer $\mathbb{Z}_q$[4], hence $q = m$[5], there are no exceptional orbits.

2.3 Singular Orbits

Since $\mathbb{S}^1$ is a subgroup of $\mathrm{SU}(2)$, which induces the Hamiltonian torus action, hence must have a fixed points, whose stablizer containing $\mathbb{S}^1$ as a subgroup, hence with dimension bigger that 1, hence the orbit will be strictly less than 3, this is a singular orbit.

The dimension of a singular orbit could only be 2,1 or 0, the dimension of the corresponding stablizer will be 1,2 or 3, since $\mathrm{SU}(2)$ has no 2-dimensional Lie subgroups[6], thus it could only be 1-dimensional Lie subgroups or $\mathrm{SU}(2)$ itself, the later case is the fixed point of this $\mathrm{SU}(2)$ action, the tubular neighbourhood has the homotopy type $\mathbb{C}^2$, and with moment map $\mu(x) = 0$.

To investigate the 1-dimensional case, we use the 2-sheeted covering map:

$$\phi : \mathrm{SU}(2) \cong \mathbb{S}^3 \longrightarrow \mathrm{SO}(3) \cong \mathbb{RP}^3$$

the later has only 2 non-conjugated 1-dimensional Lie subgroups $\mathrm{O}(2)$ and $\mathrm{SO}(2) \cong \mathbb{S}^1$, now we denoted by the H the stablizer of a singular orbit of dimension 1, that ϕ will induce a covering map on the level of orbits:

$$\bar{\phi} : \mathrm{SU}(2)/H \longrightarrow \mathrm{SO}(3)/\phi(H)$$

The later is either $\mathbb{S}^2$ or $\mathbb{RP}^2$, hence no matter what H is, this singular orbit is $\mathbb{S}^2$, hence the tubular neighborhood $(\mathrm{SU}(2) \times \mathbb{R}^2)/H$ has the homotopy type of line bundle over $\mathbb{P}^1$, i.e an $\mathcal{O}(m)$ for some m odd.

So now, we can see that the quotient space $M/\mathrm{SU}(2)$ is a 1-dimensional closed manifold with boundary, i.e. a closed interval $[a, b]$ with each end point corresponds with a singular orbit, but we didn't know what singular orbits are, both fixed points? Both $\mathbb{S}^2$? Or one for each other? We will use the following lemma to except the first case.

Lemma 2. *The function $f = \frac{1}{2} \parallel \mu \parallel^2 : M \longrightarrow [0, c]$ can be viewed as the quotient map, i,e. there exists a bijection $g : M/\mathrm{SU}(2) \longrightarrow [0, c]$[7], such that the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & [0,c] \\ \downarrow \pi & \nearrow g & \\ [a,b] \cong M/\mathrm{SU}(2) & & \end{array}$$

Proof:One can consider the descending of f, namely

$$g : M/\mathrm{SU}(2) \longrightarrow [0, c]$$

$$[x] \mapsto \frac{1}{2} \parallel \mu(x) \parallel^2$$

Notice that this is well-defined, since for $A \in \mathrm{SU}(2)$, we have $\mu(Ax) = \mu(x)$, we can see it from the commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\mu} & \mathfrak{su}(2)^* \\ \downarrow A & & \downarrow \mathrm{Ad}^*_A \\ M & \xrightarrow{\mu} & \mathfrak{su}(2)^* \end{array}$$

[3]A normal bundle in a orientable manifold is always orientable, now our symplectic 4-manifolds is orientable, thus the action must be trivial.

[4]The action of $\mathrm{SU}(2)$ on this tubular neighbourhood is

$$\mathrm{SU}(2) \times \left((\mathrm{SU}(2) \times \mathbb{R})/\mathbb{Z}_q\right) \longrightarrow (\mathrm{SU}(2) \times \mathbb{R})/\mathbb{Z}_q$$

$$(A, [B, r]) \mapsto [AB, r]$$

and if $[AB, r] = [B, r]$ which implies that there exists $U \in \mathbb{Z}_q$ such that $AB = UB$, hence the stablizer of $[B, r]$ is all $U \in \mathbb{Z}_q$, which is $\mathbb{Z}_q$.

[5]Notice that, not as the case of $\mathbb{S}^1$, the cyclic subgroups $\mathbb{Z}_q, \mathbb{Z}_m$ of $\mathrm{SU}(2)$ are not conjugate.

[6]The connected Lie subgroups of a Lie group corresponds to a Lie subalgebra of its Lie algebra, since $\mathfrak{su}(2) \cong \mathrm{R}^3$, it has no 2-dimensional Lie subalgebras.

[7]This lemma actually gives a proof of the quotient space is a closed interval!

we have $\mu(Ax) = \mathrm{Ad}^*_A \mu(x)$, for any $X \in \mathfrak{su}(2)$, we have

$$\langle \mathrm{Ad}^*_A \mu(x), X \rangle = \langle \mu(x), A^* X A \rangle = \frac{i}{2}\mathrm{tr}(A^* X A \mu(x))$$
$$= \frac{i}{2}\mathrm{tr}(X\mu(x)) = \langle \mu(x), X \rangle$$

Now, it suffices to show that g is bijective, it is surjective since f, π are surjective, differentiate f we have

$$(df)_x Y = \langle \mu(x), (d\mu)_x Y \rangle$$

where $\langle , \rangle$ is the inner product on $\mathfrak{su}(2)^*$, now, if $x \in M$ is a critical point of f, then for $Y \in \ker(df)_x$ if and only if $(d\mu)_x Y \in T_{\mu(x)}\mathbb{S}^2 \cong T_{\mu(x)}(\mathrm{SU}(2).\mu(x))$, which only if x is in a singular orbit, thus after the descending g is an injective, hence leads our result. ■

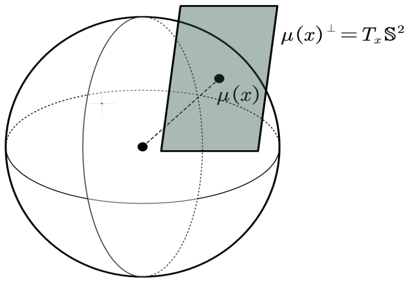


Figure 1: figure

Now to deduce the classification, we assume $f(M) = [a, b]$, we choose $c \in [a, b]$, then $f^{-1}(c, b]$ is a tubular neighbourhood of a singular orbit, since $b \neq 0$, it is a line bundle $\mathcal{O}(m)$ over $\mathbb{P}^1$, then as for $f^{-1}([a, c])$,there are only two cases:

1). If $a = 0$, $f^{-1}(a)$ is the fixed points, its tubular neighbourhood has the homotopy type of $\mathbb{C}^2$, then M is obtained by gluing a disk to a line bundle, which is $\mathbb{P}^2$

2).If $a > 0$, then $f^{[a,c]}$ is also a line bundle, M is obtained by gluing two bundles, it is a Hirzebruch surface.