Ex 8.9 in Ch4 $SO(n)$ (co)adjoint orbit

8. $T^2 = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & 0 \\ & 0 & \cos\varphi & -\sin\varphi \\ & & \sin\varphi & \cos\varphi \end{pmatrix} \middle| \theta, \varphi \in \mathbb{R} \right\} \subset SO(4)$

identify $so(n)^*$ with $so(n)$

$X \in so(n)^*$, $A \in so(n)$, $\langle X, A \rangle = \sum_{i,j} x_{ij} a_{ij}$

$so(4) \xrightarrow{\sim} \wedge^2 (\mathbb{R}^4)^*$

Hodge star operator $*$ acts on $\wedge^2 (\mathbb{R}^4)^*$

$\wedge^2(\mathbb{R}^4)^* = \mathrm{span}_{\mathbb{R}} \{ e_i^* \wedge e_j^* \}_{i<j}$

$\wedge_+ = \mathrm{span}_{\mathbb{R}} \{ \frac{1}{2}(e_1^* \wedge e_2^* + e_3^* \wedge e_4^*), \frac{1}{2}(e_1^* \wedge e_3^* - e_2^* \wedge e_4^*)$
$, \frac{1}{2}(e_1^* \wedge e_4^* + e_2^* \wedge e_3^* \}$

$\wedge_- = \mathrm{span}_{\mathbb{R}} \{ \frac{1}{2}(e_1^* \wedge e_2^* - e_3^* \wedge e_4^*), \frac{1}{2}(e_1^* \wedge e_3^* + e_2^* \wedge e_4^*)$
$, \frac{1}{2}(e_1^* \wedge e_4^* - e_2^* \wedge e_3^* \}$

$* : \wedge^2(\mathbb{R}^4)^* \to \wedge^2(\mathbb{R}^4)^*$

$(*\alpha) \wedge \beta = (\alpha, \beta)\, e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^*$

$$\begin{cases} *(e_1^* \wedge e_2^*) = e_3^* \wedge e_4^* \\ *(e_1^* \wedge e_3^*) = -e_2^* \wedge e_4^* \\ *(e_1^* \wedge e_4^*) = e_2^* \wedge e_3^* \end{cases}$$

$A \in so(4)$, $A = A_+ + A_-$

$A = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$ $\quad A_+ = \frac{1}{2}(a+f, b-e, c+d)$

$A_- = \frac{1}{2}(a-f, b+e, c-d)$

so the generic orbit of A is $S^2 \times S^2$

determined by $\|A_+\|^2$ and $\|A_-\|^2$

$$\|A\| = \|A_+\|^2 + \|A_-\|^2$$

$$\tfrac{1}{2}\mathrm{Pf}(A) = \|A_+\|^2 - \|A_-\|^2 = \tfrac{1}{2}(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})$$

(Pfaffin)

$$\mu : SO(4)\cdot A \hookrightarrow so(4)^* \to (t^2)^*$$

$$A \mapsto A \mapsto (a_{12}, a_{34})$$

$i: T^2 \hookrightarrow SO(4)$

$$i_* : (a, b) \mapsto \begin{pmatrix} 0 & a & & \\ -a & 0 & & \\ & & 0 & b \\ & & -b & 0 \end{pmatrix}$$

Let $\mathcal{O}$ be an orbit with invariant

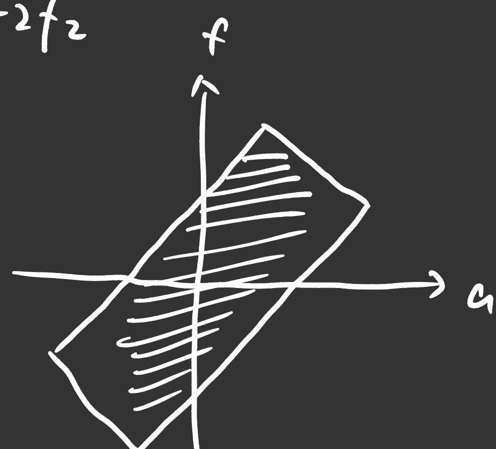
$$\|A\|^2 = f_1, \quad \mathrm{Pf}(A) = f_2$$

$$\|A_+\|^2 = f_1 + 2f_2, \quad \|A_-\|^2 = f_1 - 2f_2$$

$$A_+ = \tfrac{1}{2}(a+f,\ b-e,\ c+d)$$

$$A_- = \tfrac{1}{2}(a-f,\ b+e,\ c-d)$$

$$\mu(A) = (a, f)$$



9. (co)adjoint orbit

$$A = \left(\begin{array}{cc|c} 0 & -1 & \\ 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right) \in so(n+2)^*$$

$$SO(n+2)\cdot A \approx Q_n = \{\sum_{j=0}^{n+1} z_j^2 = 0 \text{ in } \mathbb{C}P^{n+1}\}$$

with diagonal action

$$\forall X \in SO(n+2), \quad [z_0, \cdots z_{n+1}] = [x_k + i y_k]$$

$$x = (x_0, \dots x_{n+1})^T \quad y = (y_0, \cdots y_{n+1})^T$$

$$X \cdot [z_0, \cdots z_{n+1}] = [Xx + iXy]$$

with moment map (Ch3 Ex14).

$$[z_0, \cdots z_{n+1}] \mapsto \frac{x \wedge y}{\|x\|^2 + \|y\|^2} \in so(n+2)^*$$

$$= \left(\frac{x_i y_j - x_j y_i}{\|x\|^2 + \|y\|^2}\right)_{i,j}$$

adjoint orbit of A

$= \{$skew-symmetric matrices, rank $= 2$, norm $= 1\}$

$$= SO(n+2) / SO(n) \times SO(2)$$

$$= \widetilde{G_2}(\mathbb{R}^{n+2})$$

$$= Q_n = \{\sum_{j=0}^{n+1} z_j^2 = 0 \text{ in } \mathbb{C}P^{n+1}\}$$

$$[z_k] = [x_k + i y_k] \quad \begin{cases} \sum x_k^2 = \sum y_k^2 \\ x_k y_k = 0. \end{cases}$$

$k = [\frac{n+2}{2}]$, so $(\mathbb{R}^2)^k \subset \mathbb{R}^{n+2}$

and $T^k = (SO(2))^k \hookrightarrow SO(n+2)$

$n=2$
$k=2$

$T^2 \hookrightarrow SO(4)$

so the moment map

$$SO(n+2)\cdot A \to so(n+2)^* \to (t^k)^*.$$

$$A \longmapsto (a_{12}, a_{34}, \cdots a_{2k-1,2k})$$

The fixed point

$$T^k = \begin{pmatrix} (\ \ ) & & & \\ & (\ \ ) & & \\ & & \ddots & \\ & & & (\ \ ) \end{pmatrix} \subset SO(n+2)$$

Fixed point $\begin{pmatrix} A_1 & & & \\ & \ddots & & \\ & & A_k & \\ & & & 0 \end{pmatrix}$ one of A_i is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

others $= 0$.

$\mu(A) =$ Convex hull of $\pm e_i \in \mathbb{R}^k$.

Q_n

$$\mu: [z_0, \cdots z_n] \mapsto \frac{2}{\sum |z_j|^2}(x_0 y_1 - x_1 y_0, \cdots, x_{2k-2} y_{2k-1} - x_{2k-1} y_{2k-2})$$

Let $0 < \lambda_1 < \cdots < \lambda_k$ be k real numbers.

$$f = \frac{2}{\sum |z_j|^2}(\lambda_1 (x_0 y_1 - x_1 y_0) + \cdots + \lambda_k (x_{2k-2} y_{2k-1} - x_{2k-1} y_{2k-2})).$$

Prove f is a perfect Morse function and it has

— if n is odd

	$\mathbb{Z} \to$	$0 \to$	$\mathbb{Z} \to \cdots$	$\mathbb{Z} \to$	$0 \to$	$\mathbb{Z}$
index	0	1	2	$2n-2$	$2n-1$	$2n$
HM_k	$\mathbb{Z}$	0	$\mathbb{Z}$ ---			
Betti	1	0	1	0 ...		

— if n is even

$$\mathbb{Z} \to 0 \to \mathbb{Z} \quad \cdots \quad \mathbb{Z}\oplus\mathbb{Z} \quad \to \cdots \quad 0 \to \mathbb{Z}$$

index $\quad 0 \quad 1 \quad 2 \quad n \quad 2n-1 \quad 2n$

Morse Ineq: $\quad C_k(f) \geq \beta_k(M)$

When $n=2$: $\quad 0<\lambda<\mu \quad (k=2)$

$$f = \frac{2}{\sum |z_j|^2}\left(\lambda(x_0 y_1 - x_1 y_0) + \mu(x_2 y_3 - x_3 y_2)\right)$$

$$\{ z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0 \text{ in } \mathbb{C}P^3 \}$$

$\varphi : \mathcal{U}_0 \to \mathbb{C}^3 \ (\mathbb{R}^6)$

$$[z_0, \dots, z_3] \mapsto \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \frac{z_3}{z_0}\right)$$

$$f \circ \varphi^{-1} = \frac{2(\lambda y_1 + \mu(x_2 y_3 - x_3 y_2))}{1 + x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2}$$

$$1 + z_1^2 + z_2^2 + z_3^2 = 0$$

$$\begin{cases} x_1^2 + x_2^2 + x_3^2 - y_1^2 - y_2^2 - y_3^2 - 1 = 0 = \varphi_1 \\ x_1 y_1 + x_2 y_2 + x_3 y_3 = 0 \qquad = \varphi_2 . \end{cases}$$

use Lagrangian multiplier:

$$F = f + A\varphi_1 + B\varphi_2 \quad (\mathbb{R}^6)$$

$$\begin{cases} \dfrac{\partial F}{\partial x_i} = \dfrac{\partial F}{\partial y_i} = 0 \\ \dfrac{\partial F}{\partial A} = 0 \\ \dfrac{\partial F}{\partial B} = 0 \end{cases} \Rightarrow \quad x_1 = 0,\ y_1 = \pm 1, \quad \text{other} = 0 .$$

so $[1, \pm i, 0, 0]$ are critical points of index 2.

Similarly, we know all the critical points are

$$[1, \pm i, 0, 0], \quad [0, 0, 1, i], \quad [0, 0, 1, -i]$$

index 2 $\qquad$ 4 $\qquad$ 0

$-\lambda, +\lambda$ $\qquad$ μ $\qquad$ $-\mu$

Morse complex $\quad \mathbb{Z} \to 0 \to \mathbb{Z} \oplus \mathbb{Z} \to 0 \to \mathbb{Z}$

index $\quad$ 0 $\quad$ 1 $\quad$ 2 $\quad$ 3 $\quad$ 4.