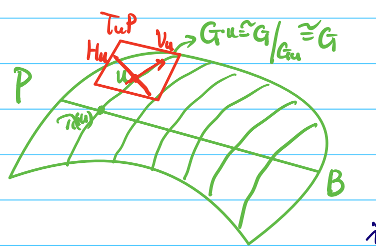
I. Connections on Principal Bundles

A principal bundle P over a base manifold B is a manifold endowed with a free G-action, s.t. the bundle projection $\pi: P \longrightarrow B$ is the orbit map.



For any $u \in P$, we have tangent map:

$$(d\pi)_u : T_uP \longrightarrow T_{\pi(u)}B.$$

we know from the previous chapter:

$$\ker(d\pi)_u = \{\underline{X}(u) \mid x \in \mathfrak{g}\} := V_u \cong \mathfrak{g}.$$

which is called the vertical direction on u, the connection on P is a choice of Horizontal subspace $H_u \subset T_uP$ for any $u \in P$, i.e:

$$T_uP = V_u \oplus H_u.$$

And we wish this choice to be smooth and G-invariant:

Def 1.1: A connection is distribution on P, $u \longmapsto H_u \subset T_uP$ s.t.:

1): The distribution is smooth,

2): $H_u \oplus V_u = T_uP$, horizontal.

3): $\forall g \in G$, $g: P \to P$, $H_{gu} = (dg)_u H_u$, G-invariant.

- The smooth distribution means, if X is a smooth vector field on P, then so is X^{H_u}.
- The condition 2) means there is a splitting. (exact sequence)

$$0 \longrightarrow V_u \hookrightarrow T_uP \twoheadrightarrow H_u \longrightarrow 0.$$

- The 3) means, the splitting is G-equivariant.

$$\begin{tikzcd}
0 \arrow[r] & V_u \arrow[r] \arrow[d, "(dg)_u"', "\cong"] & T_uP \arrow[r] \arrow[d, "(dg)_u"', "\cong"] & H_u \arrow[r] \arrow[d, "(dg)_u"', "\cong"] & 0 \\
0 \arrow[r] & V_{gu} \arrow[r] & T_{gu}P \arrow[r] & H_{gu} \arrow[r] & 0
\end{tikzcd}$$

This distribution actually gives a decomposition on T_uP, i.e $\forall x \in T_uP$,

$$X = X^{V_u} + X^{H_u},$$

however, for X^{V_u}, $\exists! A \in \mathfrak{g}$, s.t. $\underline{A}(u) = X^{V_u}$, which defines a map:

$$\omega : T_uP \longrightarrow \mathfrak{g}.$$
$$X \longmapsto A.$$

Thm 1.1: The above ω defines a $\mathfrak{g}$-valued 1-form, i.e: $\omega \in \Omega^1(P;\mathfrak{g})$ and satisfies:

1): $\omega(\underline{A}(u)) = A$

2): $g^*\omega = Ad_{g^{-1}}\omega$

Conversely, for any such $\omega \in \Omega^1(P;\mathfrak{g})$, defines a connection on P.

Sketch of the Proof: 1) is obviously.

2): By definition of pull-back: $\forall x \in T_u P$

$$g^*\omega(x) = \omega((dg)_u x) = \omega((dg)_u x^{V_u} + (dg)_u x^{H_u})$$

$$= \omega((dg)_u x^{V_u})$$

we assume $\omega(X^{V_u}) = A$, it suffices to show:

$$Ad_{g}^{-1}A = \omega((dg)_u X^{V_u})$$

$$\Longleftrightarrow$$

$$\underline{Ad_g^{-1}A}(u) = (dg)_u X^{V_u}.$$

$$\Longleftrightarrow$$

$$\underline{Ad_g^{-1}A}(u) = (dg)_u \underline{A}$$

Which can be verified hardly by definition, for the converse direction, the horizontal distribution is:

$$\mathcal{H}_u = \ker \omega .$$

Hence the connection on the principal bundle P is equivalent to the G-invariant $\mathfrak{g}$-valued 1-form $\Omega^1(P;\mathfrak{g})$.

- In the case of complex vector bundle E of rank n over a (Riemann) surface B, the connection on E is also a Lie algebra-valued 1-form, since the E is associated with the frame bundle $Fr(E)$ over B, which is a $GL_n(\mathbb{C})$-principal bundle.

Another Example: We take $S^1 \times S^3 \longrightarrow S^3 \subset \mathbb{C}^2$, by rotation, the quotient map makes it a principal bundle over $S^2 \cong \mathbb{P}^1$; $\pi: S^3 \longrightarrow S^2$. $Lie(S^1) = \mathbb{R}$, hence the connections on S^3 are the $\Omega^1(S^3)$ satisfies 2 properties:

1): $\omega(\underline{a}) = a$,

2): $\theta^*\omega = Ad_\theta \omega$.

Since S^1 is Abelian, its adjoint action on $Lie(S^1) \cong \mathbb{R}$ is trivial, it is hardly to check that 2) identically holds. 1) says that ω is invariant along the direction of $\underline{a}$, i.e. $\mathcal{L}_{\underline{a}}\omega = 0$, and $i_{\underline{a}}\omega = a$,

For example: $\alpha = \sum_{j=1}^{2} (-p_j\, dq_j + q_j\, dp_j)$, $z_i = p_i + i q_i$.

II: Curvature Form.

From the curvature form on a Vector bundle, we know that, if ω is a connection form, Ω is its Curvature, then, we have:

$$\Omega = d\omega + \omega\wedge\omega$$

which is a matrix-valued 2-form, similarly:

Def 2.1: The curvature form of $\omega \in \Omega^1(P;\mathfrak{g})$ is:

$$\Omega = d\omega + \omega\wedge\omega.$$

Similarly, we have following property:

Thm 2.1: The curvature from satisfies:

1): $\Omega_u(X,Y) = \omega([X^{Hu}, Y^{Hu}])$, $\forall X,Y \in T_uP$.

2): $g^*\Omega = Ad_{g^{-1}}\Omega$

3): $d\Omega = [\Omega, \omega]$.

Def 2.2 A flat connection is a connection $\omega \in \Omega^1(P;\mathfrak{g})$, s.t: $\Omega = 0$, i.e the Maurer-Cartan Equation:

$$d\omega + \omega\wedge\omega = 0.$$

- By 1) in Thm 2.1, flat connection means

$$\omega([X^{Hu}, Y^{Hu}]) = 0$$

i.e, the Lie bracket of $[X^{Hu}, Y^{Hu}]$ is still horizontal, i.e, by the Frobenius Thm, the distribution $u \mapsto H_u$ is integrable,

III: Gauge Group.

From now on, we may assume that the principal Bundle P is trivial, i.e: $P = B\times G$. Then the G action on $B\times G$ by trivial on B, left multiplication on G.

Def 3.1 The gauge group Γ of a principal bundle is the Bundle Automorphism group:

$$\Gamma = Aut_G(P).$$

i.e. $\tau \in \Gamma$,

$$\begin{tikzcd} P \arrow[rr, "\tau"] \arrow[dr, "\pi"'] & & P \arrow[dl, "\pi"] \\ & B & \end{tikzcd}$$

In the case of trivial bundle, $\tau(x,g) = (x, \tau(g))$, hence which defines a map.

$$P \longrightarrow \mathrm{Aut}(G) \cong G.$$
$$(x,g) \longmapsto (g \mapsto \tau(g)).$$

Hence we can identify $\Gamma = C^\infty(P;G)$, then we can define the gauge group action on the space of connections:

Def 3.2: For $\omega \in \Omega^1(P;\mathfrak{g})$ a connection on a trivial bundle $P = B \times G$, $\tau \in \Gamma$, τ acts on ω is defined by:

$$(\tau \cdot \omega)_u := \mathrm{Ad}_{\tau_u}\omega + \tau_u^{-1} d\tau_u$$

where "$\tau_u^{-1} d\tau_u$" is symbolic definition:

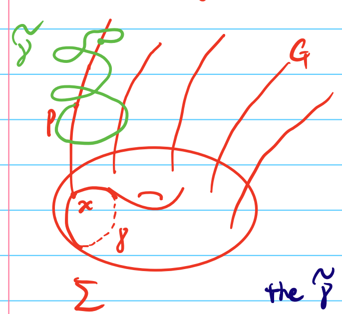
$$T_uP \xrightarrow{d\tau_u} T_{\tau_u}G \xrightarrow{\tau_u^{-1}} \mathfrak{g}.$$

So that $\tau_u^{-1} d\tau \in \Omega^1(P;\mathfrak{g})$.

Def 3.3: The moduli space of flat connections on trivial principal bundles P is:

$$\mathcal{M} = \mathcal{A}_{fl} / \Gamma.$$

IV: Holonomy.



For a loop $\gamma: S^1 \longrightarrow B$, based at x, there exists a horizontal lifting $\tilde{\gamma}: S^1 \longrightarrow P$, s.t.:

$$\begin{cases} \pi \circ \tilde{\gamma} = \gamma, \ \tilde{\gamma}(0) = p. \\ \tilde{\gamma}'^{V} \equiv 0 \end{cases}$$

i.e. the tangent vectors of the lifted curve are always horizontal.

the $\tilde{\gamma}$ start at p, but may not end at p, but still on the fibre, i.e. $\exists g \in G$, s.t.: $\tilde{\gamma}(2\pi) = g \cdot \tilde{\gamma}(0) = gp$. we denote all such $g \in G$, the holonomy group of the connection ω:

Def 4.1: $\mathrm{Hol}_\omega(B) = \{ \underbrace{g}_{=\,\mathrm{hol}_\omega(\gamma)} \in G \mid \tilde{\gamma}(2\pi) = g \cdot \tilde{\gamma}(0) \}$ is called the holonomy group of the connection ω.

There is an expression of $hol_\omega(\gamma)$ is:

$$hol_\omega(\gamma) = e^{-\int_\gamma \omega}$$

hence if ω is flat, $hol_\omega(\gamma)$ is trivial (i.e. $=1$) if γ is trivial, and for two connections, if there is a gauge transformation among them, $\tau \in \Gamma$, $\omega = \tau\cdot\omega$, then:

$$hol_{\tau\cdot\omega}(\gamma) = \tau^{-1}\, hol_\omega(\gamma)\, \tau.$$

which defines a representation:

$$\begin{aligned} \pi_1(B) &\longrightarrow G \\ [\gamma] &\longmapsto hol_\omega(\gamma). \end{aligned}$$

and two representations are conjugacy if two connections are gauge equivalent. then, we have:

$$\mathcal{A}_{fl}/\Gamma = \mathrm{Hom}(\pi_1(B), G)/G.$$