Goldman's functions.

Recall.

$\Sigma_{g,d}$ $\quad P$

$\mathcal{A} = \Omega^1(\Sigma, \mathfrak{g})$

$F: \Omega^1(\Sigma, \mathfrak{g}) \to \Omega^2(\Sigma, \mathfrak{g})$

$\mathcal{G} = C^\infty(\Sigma, G)$

$\mathfrak{g} = \mathrm{Lie}(\mathcal{G}) = C^\infty(\Sigma, \mathfrak{g})$

$\forall g \in \mathcal{G}$. $g \cdot A = g^{-1} A g + g^{-1} dg$.

Holonomy $\quad \rho: \pi_1(\Sigma, x) \to G$

$\mathcal{M} = \mathcal{A}_{\mathcal{F}1} / \mathcal{G} \quad \longleftrightarrow \quad \mathrm{Hom}(\pi_1 \Sigma, G) / \mathrm{Ad}\, G$.

Now. Ad-inv non-degenarate symmetric bilinear form

$$\mathcal{B}: \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$$

Def. $\quad \omega_A(\varphi, \psi) = \int_\Sigma \mathcal{B}_*(\varphi \wedge \psi)$

$$\wedge: \quad \Omega^*(\Sigma, \mathfrak{g}) \otimes \Omega^*(\Sigma, \mathfrak{g}) \to \Omega^*(\Sigma, \mathfrak{g} \otimes \mathfrak{g})$$
$$\mathcal{B}_*: \quad \Omega^*(\Sigma, \mathfrak{g} \otimes \mathfrak{g}) \to \Omega^*(\Sigma, \mathbb{R})$$

$\mathcal{A}$: sympleette vector space.

Ques: ω on $\mathcal{A}$ $\rightsquigarrow$ what structure on $\mathcal{M}$?

- $\mathcal{G}$-action preserves ω : $g^*\omega_A = \omega_A$. $\forall g \in \mathcal{G}$.
- Hamiltonian action, its moment map

$$\mathfrak{g} \to C^\infty(\mathcal{A})$$
$$a \mapsto H_a.$$

Hamiltonian vector field of H_a is $\underline{a}$.

$\mathfrak{g} \to C^\infty(\mathcal{A})$ is Lie algebra homomorphism.

If $\partial\Sigma = \phi$ (Atiyah-Bott)

$$H_a(A) = \int_\Sigma B_*(a \wedge F(A))$$

Now the curvature is a moment map. μ

$$\mathcal{M} = \mathcal{A}_{\mathcal{H}}/\mathcal{G} = \mu^{-1}(0)/\mathcal{G}$$

: symplectic manifold. (as long as it's a mfd)

If $\partial\Sigma \neq \phi$.

$$H_a(A) = \int_\Sigma B_*(a \wedge F(A)) + \int_{\partial\Sigma} B_*(a \wedge A)$$

Hamiltonian vector field of $H_a(A)$ is $\underline{a}$.

But $\{H_a, H_\beta\}(A) = H_{[a,\beta]}(A) + \int_{\partial\Sigma} B_*(a \wedge d\beta)$

- Central extension.

$a, \beta \in \mathfrak{g}$. $\quad c(a,\beta) = \int_{\partial\Sigma} B_*(a \wedge d\beta)$

$\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R} \quad [(a,t),(\beta,u)] := ([a,\beta], c(a,\beta))$.

Define $H_{(a,t)}(A) = H_a(A) + t$

Hence
$$\hat{\mathfrak{g}} \longrightarrow C^\infty(\mathcal{A})$$
$$(a,t) \mapsto H_{(a,t)}.$$

Now we describe the moment map.

Prop. ($\hat{\mathfrak{g}}^*$)

The pairing $\Omega^2(\Sigma, \mathfrak{g}) \oplus \Omega^1(\partial\Sigma, \mathfrak{g}) \oplus \mathbb{R}$ with $\hat{\mathfrak{g}}$ by

$$(R, \varphi, z) \otimes (a, t) \mapsto \int_\Sigma B_*(a \wedge R) + \int_{\partial\Sigma} (a \wedge \varphi) + zt$$

is non-degenarate.

$$\Rightarrow \Omega^2(\Sigma, \mathfrak{g}) \oplus \Omega^1(\partial\Sigma, \mathfrak{g}) \oplus \mathbb{R} \underset{\text{subspace.}}{\hookrightarrow} \hat{\mathfrak{g}}^*$$

Cor. $\mu: \mathcal{A} \to \hat{\mathfrak{g}}^*$

$A \mapsto (F(A), A|_{\partial\Sigma}, 1)$

is a $\mathcal{G}$-equivariant moment map for $\mathcal{G}$ action.

Now consider smaller group s.t. curvature is moment map.

$$\mathcal{G}_0 = \{ g \in \mathcal{G} \mid g|_{\partial\Sigma} = 1 \} \triangleleft \mathcal{G}$$

prop. $\mathcal{G}_0$ acts on $\mathcal{A}$ with moment map $A \mapsto F(A)$

$\mathcal{M}_0 = \mathcal{A}_{fl} / \mathcal{G}_0$ is symplectic.

$$\mathcal{M} = \mathcal{A}_{fl} / \mathcal{G} = \mathcal{M}_0 / \mathcal{G}.$$

Prop. G acts on symplectic W with $\mu: W \to \mathfrak{g}^*$. The Poisson structure of W defines a Poisson structure on W/G, the symplectic leaves of which are in one-to-one correspondence, via μ, with the coadjoint orbit of G in the $\mu(W) \subset \mathfrak{g}^*$

We apply above prop to the action of $\mathcal{G}/\mathcal{G}_0$ on $\mathcal{M}_0 = \mathcal{A}_{\mathcal{H}}/\mathcal{G}_0$ it'll give Poisson structure

$$\mathcal{M}_0 / \mathcal{G}/\mathcal{G}_0 = \mathcal{A}_{\mathcal{H}}/\mathcal{G}_0 \Big/ \mathcal{G}/\mathcal{G}_0 = \mathcal{A}_{\mathcal{H}}/\mathcal{G} = \mathcal{M}.$$

There is an exact sequence.

$$1 \to \mathcal{G}_0 \to \mathcal{G} \to \underline{\Omega^0(\partial\Sigma, G)}$$

$$\Rightarrow \mathcal{G}/\mathcal{G}_0 \subset \Omega^0(\partial\Sigma, G) = \underline{\bigoplus_{C_i \in \pi^0(\partial\Sigma)} \Omega^0(C_i; G)}$$

$$\mathrm{Lie}(\mathcal{G}/\mathcal{G}_0) \subset \bigoplus \Omega^0(C_i; \mathfrak{g})$$

We identify $\bigoplus \Omega^1(C_i, \mathfrak{g})$ with $\mathrm{Lie}(\mathcal{G}/\mathcal{G}_0)^*$ by

$$\langle (\varphi_1, \dots \varphi_d), (a_1, \dots a_d) \rangle = \sum_{i=1}^{d} \int_{C_i} B_*(\varphi_i \wedge a_i)$$

The moment map of $\mathcal{G}/\mathcal{G}_0$ acting on $\mathcal{M}_0$ is

$$\mathcal{M}_0 \to \left(\bigoplus \Omega^1(C_i; \mathfrak{g})\right) \oplus \mathbb{R}$$

$$A \mapsto \left((A|_{C_1}, \dots A|_{C_d}), 1\right)$$

Identify C_i with S^1, $\Omega^0(C_i; \mathfrak{g}) \cong \underline{\Omega^0(S^1; \mathfrak{g})}$

is Lie algebra $L\mathfrak{g}$ of the loop group

$$LG = C^\infty(S^1, G) = C^\infty(C_i; G)$$

· The coadjoint action of LG on the dual $\hat{L\mathfrak{g}}^*$.

$\Rightarrow$ Prop. $\Omega^1(S^1, \mathfrak{g}) \oplus \mathbb{1}$ can be embedded as a supspace in $\hat{L\mathfrak{g}}^*$:

$$\Omega^1(S^1, \mathfrak{g}) \to G.$$

form $\mapsto$ holonomy along the circle.

$$\left\{ \begin{array}{c} \text{coadjoint orbits} \\ \text{of } LG \end{array} \right\} \leftrightarrow \{ \text{conjugacy classes} \}.$$

Final. (Thm 2.1) ω on $\mathcal{A}$ defines a Poisson structure on $\mathcal{M}$. the symplectic leaves of which are obtained by fixing the conjugacy classes of the holonomies along the component of $\partial\Sigma$.

Eg. (S^1-bundle)

$$M = A_H/\mathcal{G} = \mathrm{Hom}(\pi_1 \Sigma_{g,d}, S^1)$$

$$\pi_1(\Sigma_{g,d}) = \langle \underbrace{\alpha_1, \dots \alpha_g, \beta_1, \dots \beta_g, \mu_1, \dots \mu_d}_{2g+d} \mid \underline{\prod \mu_i \prod \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}} \rangle$$

$$\mathrm{Hom}(\pi_1 \Sigma_{g,d}, S^1) = \begin{cases} (S^1)^{2g} & d = 0 \\ (S^1)^{2g} \times (S^1)^{d-1} & d \geq 1 \end{cases}$$

Symplectic foliation $\quad c_1, \dots c_{d-1} \in (S^1)^{d-1}$.

leaves are

$$(S^1)^{2g} \times \{c_1, \dots, c_{d-1}\}$$

Eg. ($SU(2)$-bundle) $\mathcal{M}_{0,3}$ $\mathcal{M}_{1,1}$

$$\mathcal{M}_{0,3} = \{(A,B,C) \in (SU(2))^3 \mid ABC = 1\} / \mathrm{Ad}\, G$$
$$\simeq G \times G / \underline{\mathrm{Ad}\, G}$$

$$\mathcal{M}_{1,1} = \{(A,B,C) \in (SU(2))^3 \mid ABA^{-1}B^{-1}C = 1\} / \mathrm{Ad}\, G$$
$$\simeq G \times G / \underline{\mathrm{Ad}\, G}$$

$\dim \mathcal{M}_{1,1} = \dim C$

$\dim G - \dim C.$

(Ex 1)

$\mathcal{M}_{0,3}$: $M_1, M_2, M_3 \in SU(2)$

— $M_1 M_2 M_3 = 1$

— there is no line in $\mathbb{C}^2$ which is invariant under M_1, M_2, M_3.

Consider two linear maps $f, g : \mathfrak{g} \to \mathfrak{g}$

$$f(X) = M_1 X M_1^{-1} - X, \qquad g(X) = M_3^{-1} X M_3 - X$$

and prove $\dim \operatorname{Ker} f + \dim \operatorname{Ker} g \leq \dim \mathfrak{g}$.

Let C_1, C_2, C_3 be the conjugacy classes in G of M_1, M_2, M_3.

consider
$$h : C_1 \times C_2 \times C_3 \to G$$
$$(A_1, A_2, A_3) \mapsto A_1 A_2 A_3.$$

$$T_{(M_1, M_2, M_3)} h([X, M_1], [Y, M_2], [Z, M_3])$$

$$= M_1 (Y - X) M_1^{-1} - (Y - X) + M_3^{-1} (Z - Y) M_3 - (Z - Y)$$

Deduce that h has maximal rank at (M_1, M_2, M_3) and

$$\dim \mathcal{M}_{0,3}^{C_1, C_2, C_3} = \sum \dim C_i - 2 \dim G. = \dim G - 3\, \mathrm{rk}\, G$$

$$\dim \mathcal{M}_{1,1} = \dim C$$

$$= \dim G - \mathrm{rk}\, G$$

For generic conjugacy class, it has the form G/T
$\dim C = \dim G - \mathrm{rk}\, G$.

Goldman's function.
For a Ad-invariant function $f: G \to \mathbb{R}$ and a simple closed curve C on Σ.

$$f_C : \mathcal{M} \to \mathbb{R}$$
$$[\rho] \mapsto f(\rho(C))$$

$\rho : \pi_1 \Sigma \to G$

For C a simple closed curve, let $\mathcal{G}_C = \{ g \in \mathcal{G} \mid g|_C = 1 \} \triangleleft \mathcal{G}$.
Call $\mathcal{M}_C = \mathcal{A}_{fl} / \mathcal{G}_C$

$$\mu_C : \mathcal{M}_C \longrightarrow \hat{L\mathfrak{g}}^*$$
$$A \mapsto (A|_C, 1)$$

is the moment map for the $\mathcal{G}/\mathcal{G}_C$ action on $\mathcal{M}_C$.
and thus defines a map

$$\Phi_C : \mathcal{M} \to \hat{L\mathfrak{g}}^* / LG \qquad (C \sim S^1)$$

$$A \mapsto \text{holonomy along } C.$$

The flow of a function f_C will preserve the levels of Φ_C.

Prop. Let C be a simple closed curve on Σ. The flow of f_C acts on the class $[A] \in \mathcal{M}$ of a flat connection by some element of the stabilizer $G_m(\Phi_C([A]))$ of its holonomy along C.

Goldman has proved two beautiful thm:

① compute the $\{f_{C_1}, g_{C_2}\}$ in terms of $C_1 \cdot C_2$.

② Give an explicit formula for the flow f_C.

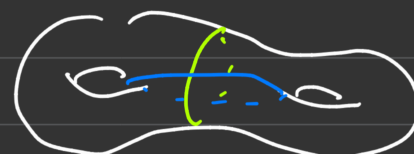
Thm 1. C_1, C_2 be two disjoint simple closed curve.

$\forall f, g$

$$\{f_{C_1}, g_{C_2}\} = 0.$$

Cor. C: simple closed curve

$$\{f_C, g_C\} = 0$$



Integrable system.

Assume $d \geq 1$ if $g=1$

$d \geq 3$ if $g = 0$.

The the maximal number of disjoint curves on Σ is $3g-3+d$.

Trinion decomposition

Goldman gives $(3g-3+d)$ rk G commuting functions on $\mathcal{M}$.

number of Ad-action invariant.

If $\partial\Sigma = \phi$.

$$\dim \mathcal{M}_{g,0} = (2g-2)\dim G \Big/ (3g-3) \operatorname{rk} G = 2$$

(simple). $\dim G = 3 \operatorname{rk} G$. $SU(2)$

If $\partial\Sigma \neq \phi$.

$\mathcal{M}_{g,d}(SU(2))$ $\rho: \pi_1\Sigma \to SU(2)$.

Let $M_1, \dots, M_d$ be matrices corresponding to the d boundary components of Σ.

$$M_j \sim \begin{pmatrix} e^{i\theta_j} & \\ & e^{-i\theta_j} \end{pmatrix}$$

Prop. Let $\theta_1, \dots, \theta_d \in [0, \pi]$ s.t.

$$\forall \varepsilon_j \in \{\pm 1\}, \quad \sum \varepsilon_i \theta_i \not\equiv 0 \pmod{2\pi}$$

Then the symplectic leaf defined by the conjugacy class of

$$\begin{pmatrix} e^{i\theta_1} & \\ & e^{-i\theta_1} \end{pmatrix}, \dots, \begin{pmatrix} e^{i\theta_d} & \\ & e^{-i\theta_d} \end{pmatrix}$$

consists of regular points of $\mathcal{M}_{g,d}(SU(2))$ and is a smooth compact symplectic manifold of dim $6g+2d-6$.

$SU(2)$. $f: SU(2) \to \mathbb{R}$. The only invariant function is the trace

Prop. All the orbits of the flow of f_C are periodic.

Now we normalize so that this periodicity corresponds to an S^1-action.

$$h_C : \mathcal{M} \to [0, 1]$$

$$h_C([\rho]) = \frac{1}{\pi} \arccos\left(\frac{1}{2} \operatorname{tr} \rho(C)\right)$$

$$\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \xrightarrow{\frac{1}{2}\operatorname{tr}} \frac{e^{i\theta} + e^{-i\theta}}{2} \xrightarrow{\arccos} \theta \to \frac{\theta}{\pi}$$

Let $\mathcal{U}_C \underset{\text{open}}{\subset} \mathcal{M} = \{\rho : \pi_1\Sigma \to SU(2) \mid \rho(C) \neq \pm \mathbb{1}\}$

$h_C(\mathcal{U}_C) \subset (0, 1)$

h_C is smooth on $\mathcal{U}_C$.

Prop. The Hamiltonian $h_C : \mathcal{U}_C \to \mathbb{R}$ is periodic.

Sketch of the pf. S^1-action.

If C doesn't disconnect the surface

$$(z \cdot \rho)(\gamma) = \begin{cases} \rho(\gamma) & \text{if } \gamma \cap C = \phi \\ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \rho(\gamma) & \text{if } \gamma \cdot C = 1. \end{cases}$$

If $\Sigma \setminus C$ has two component Σ_1 and Σ_2

$$(z \cdot \rho)(\gamma) = \begin{cases} \rho(\gamma) & \text{if } \gamma \text{ is homotopic to a curve in } \Sigma_1 \\ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \rho(\gamma) \begin{pmatrix} \bar{z} & 0 \\ 0 & z \end{pmatrix} & \text{if } \gamma \text{ is homotopic to a curve in } \Sigma_2. \end{cases}$$