Universal bundle and classifying space

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We shall consider numerable coverings, that is, locally finite numerable coverings $\{U_i\}_{i=1}^{\infty}$ such that there exists a partition of unity $\{u_i\}_{i=1}^{\infty}$ with:

$$0 \leq u_i \leq 1, \quad \overline{u_i^{-1}(]0,1])} \subset U_i, \quad \forall i$$

Notice, for instance, that any paracompact space has a numerable covering.

A fiber bundle over B is said to be numerable if there exists a numerable covering of B which makes it locally trivial. In this note, all the fiber bundles will be assumed to be numerable, even if I forget to mention it explicitly.

定义 1 (Principal G-bundle) Let $p : E \to B$ be a (G,F)-bundle, F is the fiber, G is the structure group, if $F = \mathrm{G}$ and satisfying the following properties.

(1) The total space E has a free, fiberwise right G action.
(2) The induced action on fibers is free and transitive.
(3) There exist local trivializations

$$\psi : U \times \mathrm{G} \to p^{-1}(U)$$

that are equivariant.

$$\psi(x, g)h = \psi(x, gh)$$

we say $p : E \to B$ is a principal G-bundle.

定义 2 (Pull-back bundle) Let $p : E \to B$ be a fiber bundle, and let $f : B' \to B$ be a continuous map, define the pull-back bundle over B' as follows:

$$f^*(E) = \{(b', e) : f(b') = p(e)\}$$

$$\begin{array}{ccc} f^*(E) & \dashrightarrow & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array}$$

命题 1 *Let $p : E \to B$ be a fiber bundle, then the pull-back bundle $f^*(E)$ is a fiber bundle with the same fiber. Moreover, if $p : E \to B$ is a principle* G*-bundle, then $f^*(E)$ is a principle* G*-bundle as well.*

证明 Let $\varphi_\alpha : U_\alpha \times \mathrm{G} \to p^{-1}(U_\alpha)$ be local trivializations of the bundle $p : E \to B$, for $p' : f^*E \to B'$, we define

$$\begin{aligned} \varphi'_\alpha : f^{-1}U_\alpha \times \mathrm{G} &\longrightarrow p'^{-1}(f^{-1}U_\alpha) \\ (b', g) &\longmapsto (b', \varphi_\alpha(f(b'), g)) \end{aligned}$$

then φ'_α are local trivializations of the bundle $p' : f^*E \to B'$, and if $\varphi_\alpha(x, g)h = \varphi_\alpha(x, gh)$, we have

$$\begin{aligned} \varphi'_\alpha(b', g)h = (b', \varphi_\alpha(f(b'), g))h &\xlongequal{fiberwise} (b', \varphi_\alpha(f(b'), g)h) \\ &= (b', \varphi_\alpha(f(b'), gh)) = \varphi'_\alpha(b', gh) \end{aligned}$$

so $p' : f^*E \to B'$ is a principle G-bundle. □

定理 1 *Let $p : E \to B$ be a principle* G*-bundle, and let $f_0 : X \to B$ and $f_1 : X \to B$ be homotopic maps.Then the pull - back bundles are isomorphic,*

$$f_0^*(E) \cong f_1^*(E)$$

The main step in the proof of this theorem is the basic *Covering Homotopy Theorem* for fiber bundles which we now state.

引理 1 (Covering Homotopy Theorem) *Let $p : E \to B$ and $q : Z \to Y$ be fiber bundles with the same fiber F, where B is normal and locally compact. Let h_0 be a bundle map*

$$\begin{array}{ccc} E & \xrightarrow{\tilde{h_0}} & Z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{h_0} & Y \end{array}$$

Let $H : B \times I \to Y$ be a homotopy of h_0 (i.e. $h_0 = H|_{B\times\{0\}}$). Then there exists a covering of the homotopy H by a bundle map

$$\begin{array}{ccc} E \times I & \xrightarrow{\tilde{H}} & Z \\ {\scriptstyle p\times 1}\downarrow & & \downarrow{\scriptstyle q} \\ B \times I & \xrightarrow{H} & Y \end{array}$$

Proof of the theorem 1. Let $p : E \to B$, and $f_0; X \to B$ and $f_1 : X \to B$ be as in the statement of the theorem. Let $H : X \times I \to B$ be a homotopy with $H_0 = f_0$ and $H_1 = f_1$. Since there is a bundle map

$$\begin{array}{ccc} f_0^*E & \longrightarrow & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & B \end{array}$$

so by the covering homotopy theorem there is a covering of the homotopy H

$$\begin{array}{ccc} f_0^*E \times I & \xrightarrow{\tilde{H}} & E \\ {\scriptstyle p'\times 1}\downarrow & & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{H} & B \end{array}$$

It's a bundle morphism with the same fiber F, so (by Lemma 2) there is a bundle isomorphism between $f_0^*E \times I$ and H^*E

$$\begin{array}{ccccc} f_0^*E \times I & \xrightarrow{\cong} & H^*E & \dashrightarrow & E \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{=} & X \times I & \overset{H}{\dashrightarrow} & B \end{array}$$

We know $H_1 = f_1$, so restricting this isomorphism to $X \times \{1\}$, we have $f_0^*E \cong f_1^*E$. ☐

引理 2 *Let $E_1 \to B_1$ and $E_2 \to B_2$ be two (G,F)-bundles. Any morphism*

$$\begin{array}{ccc} E_2 & \xrightarrow{g} & E_1 \\ \downarrow & & \downarrow \\ B_2 & \xrightarrow{f} & B_1 \end{array}$$

induces an isomorphism $g' : E_2 \to f^*E_1$.

证明 We define the map $g' : E_2 \to f^*E_1$ as $e_2 \mapsto (p(e_2), g(e_2))$, then g' is a bundle morphism. It's easy to check that g' is injective: if $g'(e) = g'(e')$, then $p(e) = p(e')$ and $g(e) = g(e')$, so $e' = x \cdot e$ for some $x \in G$. Then we have $g(e) = g(x \cdot e)$, one sees that $x = 1$.(Since g is a isomorphism in each fiber.) Moreover g' is surjective as well: for some $b_2 \in B_2$, we know if restrict the three bundles f^*E_1, E_2 *and* E_1 on b_2, b_2 *and* $f(b_2)$ respectively, they are isomorphic. So for some $(b_2, e_1) \in f^*E_1$, we have some $e' \in E_2, p(e') = b_2$, and we know exist some $x \in G$, $g(x \cdot e') = e_1$. Then we have $x \cdot e' \in E_2, g'(xe') = (p(xe'), g(xe')) = (b_2, e_1)$. ☐

We try now to construct a principal G-bundle which is universal. In other words, we want a bundle $\mathcal{E} \to \mathcal{B}$ such that any principal G-bundle is induced by a map $B \to \mathcal{B}$. More precisely, we define the universal bundles as follows.

定义 3 (Universal bundles) A numerable principal G-bundle $\mathcal{E} \to \mathcal{B}$ is called universal, if

(1) For any numerable principal G-bundle $E \to B$, there exists a map $f : B \to \mathcal{B}$ such that E is isomorphic to $f^*\mathcal{E}$.
(2) Two maps $f, g : B \to \mathcal{B}$ induce isomorphic bundles *if and only if* they are homotopic.

注 A principal G-bundle $\mathcal{E} \to \mathcal{B}$ is called universal if the pull back construction

$$\rho : [B, \mathcal{B}] \to Prin_{\mathrm{G}}(B)$$

is a bijection for every space B. Here $[B, \mathcal{B}]$ is the set of the whole homotopic classes, and $Prin_{\mathrm{G}}(B)$ is the set of the whole isomorphic classes of principal G-bundles over B. The two conditions in the definition guarantee that this map is well-defined and bijective. $\mathcal{E}$ and $\mathcal{B}$ are called the universal bundle and classifying space of G respectively, one can denote them as EG and BG sometimes.

The main goal of this note is to prove that universal bundles exist for every group G, and that the classifying spaces are unique up to homotopy type. Milnor has given a very beautiful (and, furthermore, explicit) construction of universal bundles, called the Milnor join.

定义 4 (Milnor join) Let G be a group and consider the join

$$G^{*(n+1)} = G * G * \cdots * G = G^{n+1} \times \Delta^n / \sim$$

Here $G^{n+1} \times \Delta^n = \{(x_0, t_0; x_1, t_1; \cdots ; x_n, t_n) | x_i \in G, t_i \in [0, 1], \sum t_i = 1\}$ and

$$(x, t) \sim (x', t') \iff t = t' \ and \ x_i = x'_i \ (when \ t_i = t'_i \neq 0)$$

We shall write $\langle x_o, t_0; \cdots ; x_n, t_n \rangle$ for the equivalence class of the element under consideration. The limit

$$\mathcal{J}(G) = \lim_{n \to \infty} G^{*(n+1)}$$

is defined by the inclusion maps $G^{*(n+1)} \to G^{*(n+2)}$. Any element in $\mathcal{J}(G)$ will thus be written $\langle x_o, t_0; \cdots ; x_n, t_n; \cdots \rangle$ or for brevity $\langle x, t \rangle$. Notice that, for any element of $\mathcal{J}(G)$, all the t_i's, except a finite number, are zero. We endow this set with the least expensive topology such that all the maps

$$t_i : \mathcal{J}(G) \longrightarrow [0, 1]$$
$$\langle x_o, t_0; \cdots ; x_n, t_n \rangle \longmapsto t_i$$

and

$$t_i^{-1}(]0,1]) \longrightarrow G$$
$$\langle x_o, t_0; \cdots ; x_n, t_n\rangle \longmapsto x_i$$

are continuous.

Notice that the group G acts on $G^{*(n+1)}$ and $\mathcal{J}(G)$ by

$$g \cdot \langle x, t\rangle = \langle gx, t\rangle$$

which is clearly free actions. So we get a principal G-bundle

$$p : \mathcal{J}(G) \longrightarrow \mathcal{J}(G)/G$$

Since G does not act on the coordinate t_i, there is also a continuous map

$$\widetilde{t_i} : \mathcal{J}(G)/G \longrightarrow [0,1]$$

It turns out that the open subsets $V_i = \widetilde{t_i}^{-1}(]0,1])$ constitute a numerable covering of $\mathcal{J}(G)/G$ and that the principal bundle $\mathcal{J}(G) \to \mathcal{J}(G)/G$ is trivialized on these open sets.

We shall now explain (without too much proof) that the principal bundle $\mathcal{J}(G) \to \mathcal{J}(G)/G$ were made to be universal.

命题 2 *For any numerable principal G-bundle, E over B, there exists a map $f : B \to \mathcal{J}(G)/G$ such that $f^*\mathcal{J}(G)$ is isomorphic with E.*

证明 We construct two maps f and g such that the diagram

$$\begin{array}{ccc} E & \overset{g}{\dashrightarrow} & \mathcal{J}(G) \\ {\scriptstyle p}\downarrow & & \downarrow \\ B & \overset{f}{\dashrightarrow} & \mathcal{J}(G)/G \end{array}$$

commutes. For that, we use a partition of unity $\{u_n\}$ on B such that $E|_{u_n^{-1}(]0,1])}$ is trivial. Put $U_n = u_n^{-1}(]0,1])$, let

$$\begin{array}{ccc} U_n \times G & \xrightarrow{h_n} & E|_{U_n} \\ & \searrow \quad \swarrow & \\ & U_n & \end{array}$$

be a local trivialization, and call q_n the projection $U_n \times G \to G$. The map g we want to define is simply

$$g(z) = \langle q_0 h_0^{-1}(z), u_0(p(z)); \cdots ; q_n h_n^{-1}(z), u_n(p(z)); \cdots \rangle$$

This is better defined than one might think: of course, $h_n^{-1}(z)$ is defined only for $z \in p^{-1}(U_n)$, but, if this is not the case, $u_n(p(z)) = 0$.

Now g is a bundle morphism, we know $E \cong f^*\mathcal{J}(G)$ by Lemma 2. □

It is slightly more technical, but not much more difficult, to prove that the Milnor join is indeed universal. We need the next proposition.

命题 3 *Let $f : X \to \mathcal{J}(G)/G$ and $g : X \to \mathcal{J}(G)/G$ be continuous maps, and there is an bundle isomorphism*

$$\Phi : f^*\mathcal{J}(G) \longrightarrow g^*\mathcal{J}(G)$$

then f and g are homotopic.

We have now find a universal principal G-bundle $\mathcal{J}(G) \to \mathcal{J}(G)/G$ for every group G. Moreover , we will see it's unique in the meaning of homotopy equivalence.

定理 2 *Let $E_1 \to B_1$ and $E_2 \to B_2$ be universal principal G - bundles. Then there is a bundle map*

$$\begin{array}{ccc} E_2 & \xrightarrow{\tilde{f}} & E_1 \\ \downarrow & & \downarrow \\ B_2 & \xrightarrow{f} & B_1 \end{array}$$

so that f is a homotopy equivalence.

证明 Since $E_2 \to B_2$ is universal, so for principal G-bundle $E_1 \to B_1$, there is a “classifying map” $g : B_1 \to B_2$ and an isomorphism $E_1 \cong g^*E_2$. Similarly, using the universal property of $E_1 \to B_1$, we get a "classifying map" $f : B_2 \to B_1$ and an isomorphism $E_2 \cong f^*E_1$. So $E_2 \cong f^*(E_1) \cong f^*(g^*(E_2)) = (g \circ f)^*E_2$, that is $id^*E_2 \cong (g \circ f)^*E_2$, so $(g \circ f) \simeq id_{E_2}$. Similarly, $(f \circ g) \simeq id_{E_1}$, so f is a homotopy equivalence. □

We now denote $\mathcal{J}(G)$ as EG, $\mathcal{J}(G)/G$ as BG. In the same mood, one proves the following useful result.

命题 4 *Let $E \to B$ be any numerable principal G-bundle such that the total space E is contractible, then this is a universal principal G-bundle.*

We just admit the general fact that the spaces $\mathcal{J}(G)$ we have described are indeed contractible.