

Algebraic Geometry

Theorem: $\{\text{Algebraic set of } A_k^n\} \xrightleftharpoons[\overline{Z(\cdot)}]{I(\cdot)} \{\text{radical ideal of } A\}$.

① For any subset Y of A_k^n , $\overline{Z(I(Y))} = Y$,

② For any ideal $\mathfrak{a}$ of A , we have $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

↑ (Hilbert Nullstellensatz)

Thm: Any maximal ideal of A , has the form:

(Weak one) $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_i \in k$.

Thm: (Hilbert Nullstellensatz), K is a field, L/K is a field extension, which is a finitely generated as a K -algebra, then L is algebraic over K .

L is f.g. K -algebra $\Leftrightarrow \exists$ epimorphism:

$$K(T_1, \dots, T_m) \twoheadrightarrow L.$$

i.e., $\exists a_1, \dots, a_m \in L$, st $\forall a \in L$, has the form:

$$a = \sum_{\substack{\alpha_1, \dots, \alpha_m \\ c_{\alpha_1, \dots, \alpha_m} \in K}} c_{\alpha_1, \dots, \alpha_m} a_1^{\alpha_1} \dots a_m^{\alpha_m}$$

L/K is algebraic as field extension, that is for any $a \in L$ is algebraic over K .
which means, $f(a) = 0$, for some $f \in K(T)$, nontrivial.

Hilbert $\Rightarrow$ weak form.

Let m be a maximal ideal of $A = k[x_1, \dots, x_n]$, so A/m is a field, which is finitely generated as k -algebra, thus

(Hilbert, Nullstellensatz) $(A/m)/k$ is algebraic over k .

consider the map: $k \rightarrow A \rightarrow A/m$

$$\begin{array}{ccc} k & \xrightarrow{\cong} & A/m \\ \downarrow & & \downarrow \\ \text{algebraically,} & & k \end{array} \quad \begin{array}{l} \exists f \text{ im. } \in k[t], f(a) = 0 \\ k \text{ algebraically closed} \\ \Rightarrow f(t) = at + \beta, a \neq 0, \\ \Rightarrow a = -\frac{\beta}{\alpha} \in k. \end{array}$$

$$A/m \cong k.$$

so, we have the epimorphism: $A \xrightarrow{\pi} k$, $\ker(\pi) = m$, let $a_i = \pi(x_i)$ for each i , then, $x_i - a_i \in \ker \pi = m$, and hence $(x_1 - a_1, \dots, x_n - a_n) \subseteq m$,

it is maximal ideal, check the quotient ring.

The proof of Hilbert Nullstellensatz:

Suppose L/k is not algebraic, since L is f.g. k -algebra, we have

$$L = k[f_1, \dots, f_m]$$

for some $f_i \in L$, Let $K_i = k(f_1, \dots, f_m)$, which is a subfield of L containing k . $\Rightarrow k = K_0 \subset K_1 \subset K_2 \subset \dots \subset K_m = L$. So K_i/K_{i-1} is not algebraic for some i ,

$1 \leq i \leq m$, choose i as large as possible, i.e.:

$$\begin{array}{c} L \\ | \\ K_i \\ | \\ K_{i-1} \\ | \\ K \\ \vdots \end{array} \quad \begin{array}{c} \dots \subset K_{i-1} \subset K_i \subset K_{i+1} \subset \dots \subset L \\ \xrightarrow{\text{algebraic}} \end{array}$$

Replacing K by $K[x]$, there exists $a \in L$ transcendental over K , and $L/K(x)$ is algebraic and finite extension. Choose a $K(x)$ basis, $e_1, \dots, e_l$ of L , then, $e_i e_j = \sum_{k=1}^l \frac{a_{ijk}(x)}{b_{ijk}(x)} e_k$. for $a_{ijk}(x), b_{ijk}(x) \in K(x)$, we also have:

$$f_i = \sum_{k=1}^l \frac{c_{ik}(x)}{d_{ik}(x)}, \quad c_{ik}, d_{ik} \in K[x].$$

Any element $y \in L$, can be written as

$$y = \sum_{\substack{\alpha_1, \dots, \alpha_m \\ \in K}} \alpha_1 \dots \alpha_m f_1^{\alpha_1} \dots f_m^{\alpha_m}$$

Then:

$$y = \frac{a_0(x) + a_1(x)e_1 + \dots + a_m(x)e_m}{b(x)}$$

for some $a_i(x), b(x) \in K[x]$, and $b(x)$ is the product of b_{ijk} and d_{ik} .

Take an irreducible polynomial $p(x) \in K[x]$ not dividing each b_{ijk} and d_{ik} .

$\frac{a}{p(x)}$ is not an element of L ,

$$\frac{1}{p(x)} c_i = \frac{a_1(x)}{b(x)} e_1 + \dots + \frac{a_m(x)}{b(x)} e_m \Rightarrow \frac{1}{p(x)} = \frac{a_p(x)}{b(x)} \Rightarrow p \mid b.$$

contradiction.

$$\{\text{closed subsets of } A_K^n\} \longleftrightarrow \{\text{radical ideal of } A\}.$$

$$\text{a point} \longleftrightarrow \text{maximal ideal of } A.$$

$$\underline{\text{irreducible closed subset}} \longleftrightarrow \text{prime ideal}$$

Definition: A non-empty topological space is called irreducible, if it cannot be written into two proper subsets, $\iff$ A topological space is irreducible, if any non-empty open subset is dense.

Example 1). A_K^2 , $Z(x, y) = Z(x) \cup Z(y)$. $Z(x, y)$ is not irreducible.

2). A_K^1 is irreducible.

Suppose X is irreducible, let U be an open subset of X . Since $X = \overline{U} \cup (X \setminus U)$. However, $X \setminus U$ is a proper closed one. $\overline{U} = X$.

Is A_K^n Hausdorff?

Proposition: A closed subset Y of A_K^n is irreducible iff $I(Y)$ is prime ideal of A .

Proof: Suppose Y is irreducible, take $f, g \in A$ s.t. $f \cdot g \in I(Y)$. So, $Y \subseteq Z(fg)$ which equals to $Z(f) \cup Z(g)$. then $(Y \cap Z(f)) \cup (Y \cap Z(g)) = Y$ by the irreducibility of Y , we have $Y \cap Z(f) = Y$ or $Y \cap Z(g) = Y$ if $Y \subseteq Z(fg)$ and $f \in I(Y) \Rightarrow I(Y)$ is a prime ideal of A .

Conversely, suppose $I(Y)$ is a prime ideal of A , to see Y is irreducible, let $Y = Y_1 \cup Y_2$ be union of two proper closed subsets $Y_1, Y_2 \subset A_K^n$.

Then $I(Y) = I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$. we have $I(Y) \supset I(Y_1)$ or, $I(Y) \supset I(Y_2)$, wlog, $I(Y_1) \subset I(Y)$, $I(Y) \subset I(Y_1)$, so, $I(Y) = I(Y_1)$, and $Y = Z(I(Y)) = Z(I(Y_1)) = Y_1 \Rightarrow Y$ is irreducible.

Corollary: A_K^n is not irreducible, i.e. any open set in A_K^n is dense.

Definition: A topological space is called Noetherian, if it satisfies the descending condition for closed subsets

For any sequence, $Y_1 \supset Y_2 \supset \dots \supset Y_n \supset \dots$ of closed subsets of X , there exists $N \in \mathbb{N}$, s.t. $Y_k = Y_{k+1}$ for all $k \geq N$,

Example: $\mathbb{A}_k^n$ is a Noetherian space?

For $Y_1 \supset Y_2 \supset \dots \supset Y_n \supset \dots$ Sequence of closed subsets of $\mathbb{A}_k^n$,
we have sequence of Ideals: $(\text{of } k[x_1, \dots, x_n])$

$I(Y_1) \subset I(Y_2) \subset \dots \subset I(Y_n)$ $\downarrow$ which is noetherian, ring,
thus it is a Noetherian Space.

2). Any subset of a Noetherian topological space X is Noetherian with the restricted topology.

3). Any algebraic set in $\mathbb{A}_k^n$ is a Noetherian space.

Theorem: In a noetherian topological space X , any closed subset of X , can be expressed as a finite union of $Y = Y_1 \cup Y_2 \cup \dots \cup Y_n$ by irreducible subsets Y_i , if we require that $Y_i \not\subset Y_j$, for any $i \neq j$, then, $\{Y_1, \dots, Y_n\}$ is unique, we call each Y_i an irreducible components of Y .

Proof: "Existence", Let Σ be the set of all closed subsets of X which cannot be a finite union of irreducible closed subsets of X . Since X is noetherian, Σ has a minimal element Y_0 , then Y_0 is irreducible, so $Y_0 = Y_1 \cup Y_2$ for two proper closed subsets Y_1, Y_2 of Y_0 , then $Y_1 \notin \Sigma, Y_2 \notin \Sigma$. we have:

$$Z = Z_1 \cup Z_2 \cup \dots \cup Z_r.$$

$$W = W_1 \cup W_2 \cup \dots \cup W_s$$

for some irreducible closed subset Z_i, W_j of X , $Y_0 = Z \cup W$, which contradicts to the choosing of $Y_0 \in \Sigma$.

"Uniqueness". Let $Y = Y_1 \cup \dots \cup Y_r = Y'_1 \cup \dots \cup Y'_s$, for some irreducible

closed subsets of X , $\sigma: Y_i \not\subset Y_j, Y_i' \not\subset Y_j'$ for all i, j .

$$Y_1 \subset Y = Y_1' \cup \dots \cup Y_s'$$

$\Rightarrow Y_1 \subset Y_i'$ for some i , similarly $Y_i' \subset Y_j \Rightarrow Y_1 \subset Y_j$
a contradiction, unless $j=1$.

$$Y_1 \subset Y_i' \subset Y_1 \Rightarrow Y_1 = Y_i'$$

that ends the proof.

$$\mathbb{Z}(\underbrace{x^2 - yz}, \underbrace{xz - x}) \subset \mathbb{A}_k^3.$$

$$x^2 - yz, xz - x.$$

$$\mathbb{Z}(\underbrace{x^2 - yz}) \cap \mathbb{Z}(xz - x).$$

$$\mathbb{Z}(x^2 - yz) \cap (\mathbb{Z}(x) \cup \mathbb{Z}(z-1))$$

$$(\mathbb{Z}(x^2 - yz) \cap \mathbb{Z}(x)) \cup (\mathbb{Z}(x^2 - yz) \cap \mathbb{Z}(z-1))$$

2021. Sep. 27th.

For any algebraic set Y of A_k^n , define the affine coordinate ring

$$A(Y) = A/I(Y)$$

For any $f \in A = k[x_1, \dots, x_n]$, and $p = (a_1, \dots, a_n) \in A_k^n$, $f(p) = f(a_1, \dots, a_n) \in k$,

$f: A_k^n \rightarrow k$ is a function

If $p \in Y$, $f, f' \in A$, $f - f' \in I(Y)$, we have $f(p) = f'(p)$.

If we take: $f = f': Y \rightarrow k$. So any element in $A(Y)$, defines a function on Y .

Theorem 1#: Let Y be an algebraic set of A_k^n then:

$$\begin{aligned} (1) \quad \{ \text{closed subsets of } Y \} &\longleftrightarrow \{ \text{radical ideals of } A(Y) \} \\ \{ \text{irreducible closed subset of } Y \} &\longleftrightarrow \{ \text{prime ideals of } A(Y) \} \end{aligned}$$

$$\{ \text{points of } Y \} \longleftrightarrow \{ \text{maximal ideal of } A(Y) \}$$

(2):

Any $f \in A(Y)$, define $D_f = \{ p \in Y \mid f(p) \neq 0 \}$, a lifting of f in A , $f(p) = \tilde{f}(p)$.

Then the family $\{ D_f \}_{f \in A(Y)}$ forms a topological basis of the Zariski topology on Y .

Proof: choose a lifting $\tilde{f}$ of f in A . Let

$$D_f = \{ p \in A_k^n \mid \tilde{f}(p) \neq 0 \},$$

Then $D_f = A_k^n - Z(\tilde{f})$ is open in A_k^n . then:

$$D_f = D_{\tilde{f}} \cap Y$$

is an open subset of Y . Any open subset of Y has the form: $Y \cap U$, U is open in A_k^n , where $U = A_k^n - Z(a)$ for some ideal $a \in A$. so:

$$Y \cap U = Y - Z(a) = \bigcup_{f \in a} (Y \cap D_f) = \bigcup_{g \in D} D_g$$

where $\bar{g}$ is the image of g in $A(Y)$.

Affine plane A_k^2 ;

Theorem: Every irreducible closed subset in A_k^2 is: A_k^2 , $\emptyset$, one point, irreducible plane curve, $Z(\bar{F})$, where $\bar{F}$ is a nonconstant polynomial in $k[x, y]$ irreducible.

Proof: Let Y be an irreducible closed subset of A_k^2 . If Y is a point, or $I(Y) = 0$, then Y is of our desired type.

Otherwise: Y is not a point and $I(Y) \neq 0$, so $I(Y)$ contains a non constant polynomial of F . Since $I(Y)$ is a prime ideal, then $I(Y)$ contains some irreducible factors of F , thus we may assume F is irreducible.

Let's proof $I(Y) = \bar{F}$.

Take $G \in I(Y) - \{0\}$, we can also assume that G is irreducible, let $D(y)$ be the gcd of F, G in $k(x)[y]$, we write $D(y) = \sum_i D_i(x) y^i$, and $D_i \in k[x]$. By multiplying a polynomial in $k[x]$, we may assume all $D_i \in k[x]$, $\gcd(D_i, x) = 1$.

Then $F = Df$ for some $f \in k(x)[y]$, by Gaussian Lemma, $f \in k[x][y] = k[x, y]$. Since $\bar{F}$ is irreducible, then $\bar{F} = \lambda D$ or $D = \lambda$ for some $\lambda \in k^\times$.

Similarly, $G = Dg$ for some $g \in k[x, y]$, so $G = \mu D$ or $D = \mu$ for some $\mu \in k^\times$.

If D is a constant, then $G \in (F)$, otherwise D is a constant, so $(\bar{F}, G) = 1$ in $k(x)[y]$, so $1 = u\bar{F} + vG$, for some $u, v \in k(x)[y]$. then:

$$1 = \tilde{u}\bar{F} + \tilde{v}G$$

where $\tilde{u}, \tilde{v} \in k[x, y]$, so, $Y \subseteq Z(\bar{F}, G) \subseteq Z(1)$, Similarly, $Y \subseteq Z(e(y))$ for some $0 \neq e(y) \in k[y]$, then $Y \subseteq Z(dx, e(y)) \Rightarrow Y$ is a point, which is absurd.

Def: An irreducible closed subset of A^n is called an affine variety. A nonempty open subset of an affine variety in A^n is called a quasi-affine variety.

Def: Let $X \subseteq A^n$ be a quasi-affine variety. A function $f: X \rightarrow k$ is called a regular at $p \in X$, if there exists an open nbhd U of p in X , and $g, h \in A = k[x_1, \dots, x_n]$, such that h is nowhere zero on U , and $f = g/h$ on U . If f is regular at each point of X , we say f is a regular function on X .

Denote by $\mathcal{O}(X)$ the ring of regular functions on X .

Lemma: Any regular function on X is continuous.

Proof: Let f be a regular function, we only need to show that $f^{-1}(a)$ is closed in X for any $a \in k$. For any $p \in X$, we choose a nbhd U of p in X , and $g, h \in A$, such that h is nowhere zero on U and $f = g/h$ on U , we have:

$$\begin{aligned} f^{-1}(a) \cap U &= \{ p \in U \mid a = f(p) = \frac{g(p)}{h(p)} \} \\ &= \{ p \in U \mid (ah - g)(p) = 0 \} \\ &= U \cap Z(ah - g) \end{aligned}$$

is a closed subset of U_p , thus $f^{-1}(a)$ is a closed subset, i.e. f continuous.

Def: Let X be a quasi-affine variety. $p \in X$, Define a local ring of p on X to be the ring of germs of functions on X near p . In other words, an element of $\mathcal{O}_{p,X}$ is a pair (U, f) where U is nbhd in X , and f is a regular function on U . We identify two pairs (U, f) and (V, g) if:

$$f|_{U \cap V} = g|_{U \cap V}.$$

Actually;

$$\mathcal{O}_p(X) = \varinjlim_{U \ni p} \mathcal{O}(U).$$

Lemma: $\mathcal{O}_p(X)$ is a local ring.

Proof: Let $m_{pX} = \{f \in \mathcal{O}_p(X) \mid f(p) = 0\}$, so m_{pX} is an ideal of $\mathcal{O}_p(X)$

is represented by (U, f) for some nbhd U of p , and $f \in \mathcal{O}_p(U)$ and that $f(p) \neq 0$, let $V = U - f^{-1}(0)$, so U is a nbhd of p in U , and f is nowhere zero on V , $(V, \frac{1}{f})$ is also a local function, this proves $\mathcal{O}_p(X)$ is a local ring with ideal m_{pX} .

$$\mathcal{O}_p(X)/m_{pX} \xrightarrow[\cong]{\text{valuation}} k$$

Def: Let X be a quasi-affine variety, define the function $k(X)$ of X to be

$$k(X) = \varinjlim_U \mathcal{O}(U)$$

where U runs over nonempty open sets of X ,

An element of $k(X)$ is an equivalence class of pair (U, f) where U is a nonempty open subset of X and $f \in \mathcal{O}(U)$, and where we identify the pairs (U, f) and (V, g) if $f|_{U \cap V} = g|_{U \cap V}$, actually $k(X)$ is a field.

(U, f) represents a nonzero-closed element of $k(X)$. iff and only if $f \neq 0 \in \mathcal{O}(U)$.

Take $V = U - f^{-1}(0)$, which is nonempty, $(V, \frac{1}{f})$ represents the inverse of (U, f) .

Theorem: Let $X \subset \mathbb{A}^n$ be an affine-variety, the coordinate ring $A(X) = A/I(X)$, then:

1) $\mathcal{O}(X) = A(X)$,

2) $\mathcal{O}_p(X) \cong A(X)_{m_{pX}}$, $\forall p \in X$, where $m_{pX} = \{f \in A(X) : f(p) = 0\}$

$$\beta): K(X) = A(X).$$

Proof: the natural homomorphism

$$A = k[x_1, \dots, x_n] \longrightarrow \mathcal{O}(X).$$

'induces an injective homomorphism':

$$A(X) = A/I(X) \longleftrightarrow \mathcal{O}(X).$$

and a homomorphism:

$$A(X)_{m_{pX}} \longleftrightarrow \mathcal{O}_p(X).$$

by definition of regular functions, we have a canonical isomorphism:

$$A(X)_{m_{pX}} \cong \mathcal{O}_p(X)$$

view $\mathcal{O}_p(X), \mathcal{O}(X), A(X), A(X)_{m_{pX}}$ as subring of $K(X)$, then $\mathcal{O}(X) \subseteq \bigcap_{p \in X} \mathcal{O}_p(X).$

$$= \bigcap_{p \in X} A(X)_{m_{pX}} = A$$

k -algebraically closed field.

Definition: 1) we say two polynomial $F, G \in k[X, Y]$ are equivalent, if $F = \lambda G$ for some $\lambda \in k^*$.

2). Define the affine curve in A_k^2 to be the equivalent class of non-constant polynomial in $k[X, Y]$.

3). The degree of a curve is the degree of a defining polynomial of the curve.

4). If $F = \prod_i F_i^{l_i}$, where F_i are the irreducible factors of F . l_i is the multiplicity of F_i in F . we say F_i are the irreducible components of F with multiplicity l_i .

5). If F is irreducible. $Z(F)$ is the affine variety, in this case, let: $A(F) = k[X, Y]/(F)$, $k(F)$ = the function field of $A(F)$. For any $p \in Z(F)$, let $G_{p,F}$ be the local ring of the curve F at p .

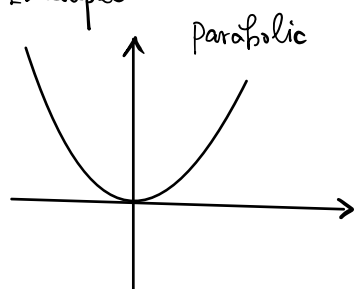
6). Let F be a curve, $p = (a, b) \in F$. p is called a simple point of F , if $\frac{\partial F}{\partial x}(p) \neq 0$ or $\frac{\partial F}{\partial y}(p) \neq 0$, in this case, the line:

$$\frac{\partial F}{\partial x}(p)(x-a) + \frac{\partial F}{\partial y}(p)(y-b) = 0$$

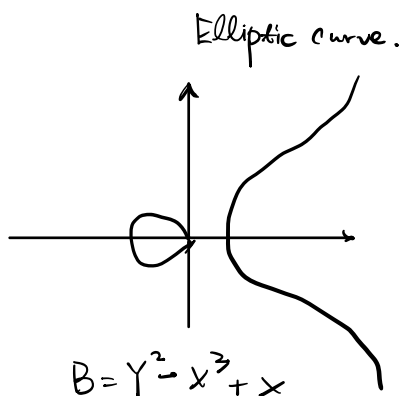
is called the tangent line to F at p , if not, we called it a multiple point.

T) A curve with only simple point is called a non singular point.

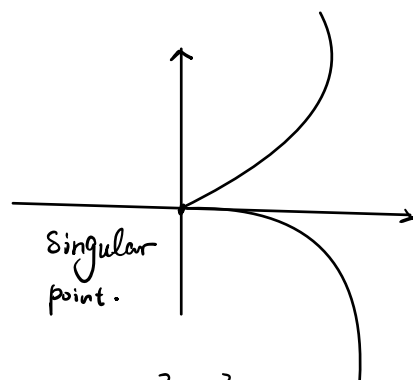
Example:



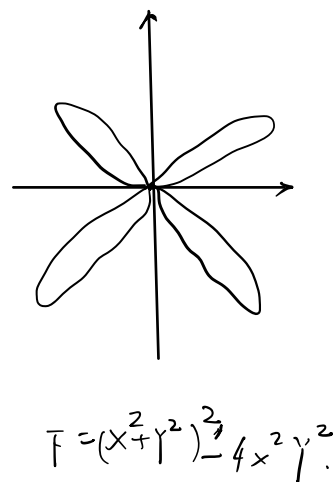
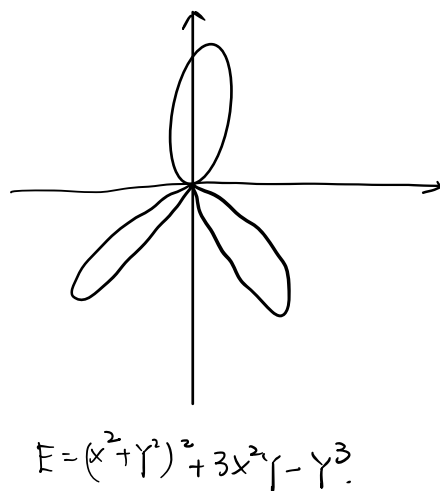
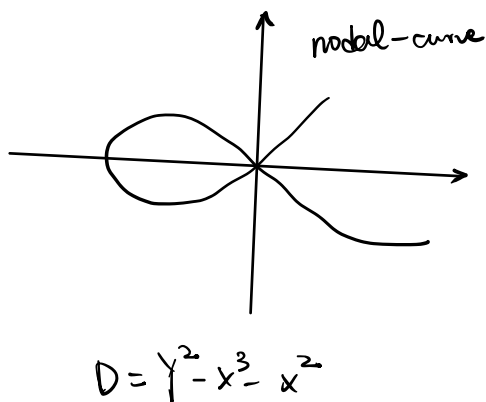
$$A = Y - X^2$$



$$B = Y^2 - X^3 + X$$



$$C = Y^2 - X^3$$



8). Let $\bar{F}$ be a curve, $p = (0,0)$, write $\bar{F} = \bar{F}_m + \dots + \bar{F}_n$, where $\bar{F}_i$ is Homogenous of degree i , $\bar{F}_m \neq 0$, $m \leq n$, then; define $m_p(\bar{F}) = m$.

$$p \text{ is a point of } \bar{F} \Leftrightarrow m_p(\bar{F}) \geq 1$$

$$p \text{ is a single point of } \bar{F} \Leftrightarrow m_p(\bar{F}) = 1.$$

write $\bar{F}_m = \prod_{i=1}^r L_i^{r_i}$ where L_i are distinct line, where r_i are positive integer; we call L_i are the tangent line to $\bar{F}$ at $p = (0,0)$, r_i is called the multiplicity of L_i .

Lemma: Let $\bar{F} = \prod_{i=1}^r L_i^{l_i}$ be the factorization of $\bar{F}$ into its polynomials, $p = (0,0)$ then

$$m_p(\bar{F}) = \sum_i l_i m_p(\bar{F}_i).$$

If L_i is a tangent line to $\bar{F}_i$ at p of multiplicity r_i , then L_i is a tangent line to $\bar{F}$ of multiplicity $\sum_i l_i r_i$.

For general point $p(a,b)$, define an isomorphism:

$$K[X,Y] \xrightarrow{\sim} K[X,Y], \bar{F} \mapsto \bar{F}(X,Y) = \bar{F}(X+a, Y+b).$$

For any affine $\bar{F}$, define $m_p(\bar{F}) = m_{(0,0)}(\bar{F})$.

Discrete valuation Ring:

Definition: Let K be a field, A discrete valuation on K is a function $v: K \rightarrow \mathbb{Z} \cup \{\infty\}$, satisfying.

1): $v(ab) = v(a) + v(b)$, for all $a, b \in K$

4) v is surjective.

2): $v(a+b) \geq \min\{v(a), v(b)\}$, $\forall a, b \in K$.

3): $v(0) = +\infty$.

Corollary: If $v(a) \neq v(b)$, then $v(a+b) = \min\{v(a), v(b)\}$.

Definition: Let $R = \{a \in K^\times \mid v(a) \geq 0\} \cup \{0\}$.

$$m = \{a \in K^\times \mid v(a) > 0\} \cup \{0\}$$

Then R is a ^{local} ring, m is the maximal ideal of R , we call R a discrete valuation ring.

An element of R with $v(t) = 1$, is called a uniformizer of R , then $m = (t)$, and each element $a \in K$ has the form $a = ut^n$ for uniformizer $u \in R^\times$ and $n \in \mathbb{Z}$, in fact, $n = v(a)$.

Proposition: Let R be a ring which is a field, the following conditions are equivalent.

- R is a Noetherian local ring and the maximal ideal is principal,
- There exists irreducible element $t \in R$ st. any non zero element $z \in R$ has the form $z = ut^n$ for some $u \in R^\times$, and $n \in \mathbb{N}$,
- R is a discrete valuation ring.

F : a non-constant polynomial in $k[x, y]$. $p = (a, b)$, a point in A_k^2 .

$$F = F_m + F_{m+1} + \dots + F_n \quad F_i \text{ homogeneous degree } i, \\ F_m \neq 0, \quad m \leq n.$$

Define $m_p(F) = m$, then

$$p \in F \iff m_p(F) \geq 1$$

if $m_p(F) = 1$, we call p is a simple point of F , in other case, F is called the tangent line to F at p .

if $m_p(F) > 1$, write $F = \prod_{i=1}^r L_i^{r_i}$ for some positive non. equivalent linear polynomials, L_i and $r_i \in \mathbb{Z}_{>0}$, we call L_i is a tangent line to F at p with multiplicity r_i .

Lemma: Let R be an integral domain (which is not a field), then the following statements are equivalent.

1): R is a discrete valuation ring.

2): R is a Noetherian local ring, whose maximal ideal is a principal ideal.

3): there exists $u \in R$ s.t. any nonzero polynomial $z \in R$ can be uniformly write as: $z = ut^n$, for some $u \in R^*$ and $n \in \mathbb{N}$.

Theorem: Let F be a plane curve and $p \in F$, then p is a simple point of F iff $\mathcal{O}_p(F)$ is a discrete valuation ring.

Appendix: localization

Let R be a ring, a subset S of R is called a multiplicative subset if

1) $S \ni 1$,

2) s.t. $\in S$, if $s_1, s_2 \in S$.

Example: Let $R = \mathbb{Z}$, $S_1 = \{p^n \mid n \in \mathbb{N}\}$, p positive integer, which is a m.s.

$$S_2 = \mathbb{Z} \setminus \{0\}.$$

Let R be a ring and S be a m.s. subset of R . Define an equivalent relation in the set

of $\frac{a}{s}$, for $a \in A, s \in S$, as follows:

Define: $\frac{a}{s} \sim \frac{b}{t} \iff \exists r \in S, \text{st } r(at - bs) = 0$.

Define $S^{-1}A$ to be the equivalent classes of $\frac{a}{s}$ for $a \in A, s \in S$.

Define: $\frac{a}{s} \pm \frac{b}{t} := \frac{at \pm bs}{st}, \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$.

Then we get a ring $S^{-1}A$, which is called the localization of A respect to S .
We have a canonical homomorphism: $\phi: A \longrightarrow S^{-1}A$, sending $a \in A$ to $\frac{a}{1} \in S^{-1}A$, which satisfy the following universal property,

For any homomorphism: $\varphi: A \longrightarrow B$, st: $\varphi(s) \in B^\times$ for any $s \in S$ there exists a unique factorization:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \phi \downarrow & \nearrow \exists! \gamma & \\ S^{-1}A & & \end{array}$$

1) Let A be a ring and f be a non-nilpotent element of A , then $\{1, f, \dots\}$ is a multiplicative subset of A .

Denote $S^{-1}A = Af = \left\{ \frac{a}{f^n} \mid a \in A, n \in \mathbb{N} \right\} / \sim$,

$$\frac{a}{f^n} \sim \frac{b}{f^m} \iff \exists l \in \mathbb{N}, l(a f^m - b f^n) = 0 \text{ for some } l \in \mathbb{N}.$$

2) Let A be a ring, $\mathfrak{p}$ a prime ideal of A , then $S = A \setminus \mathfrak{p}$, is a multiplicative subset of A . Denote $S^{-1}A = A_{\mathfrak{p}}$.

Proof: Suppose p is a single point of $\bar{F}$ and L is a line through p not tangent to $\bar{F}$ at p . By the affine changing of coordinate, we may show $p = (0, 0)$, γ is then a tangent line of $\bar{F}$ at p , and $L = X$. So we only need to prove that $m_{p, \bar{F}}$ is generated by x . So $\bar{F} = Y + \text{higher degree term}$ (multiply $\bar{F}$ by scalar).

Then $\bar{F} = YG + X^2H$ for some $G \in k[x, y], H \in k[x, y]$, so, $YG = X^2H \in \mathcal{O}_{p, \bar{F}}$.

As $G = 1 + \text{higher term}$, then $g \in \mathcal{O}_{p, \bar{F}}^\times$, and hence $y = x^2hg^{-1} \in (x)$, thus

$$m_{p, \bar{F}} = (x, y) = (x).$$

Conversely:

Theorem 2: Let p be a point of an irreducible curve, then

$$m_p(F) = \dim_k m_{p,F}^n / m_{p,F}^{n+1} \quad \text{for } n \gg 0,$$

Proof: write $\mathcal{O} = \mathcal{O}_p(F)$, $m = m_{p,F}$,

$$0 \rightarrow m^n / m^{n+1} \rightarrow \mathcal{O} / m^{n+1} \rightarrow \mathcal{O} / m^n \rightarrow 0 \quad \text{exact.}$$

To prove, it suffices to prove: $\exists S$ s.t.:

$$\dim_k (\mathcal{O} / m^n) = n \cdot m_p(F) + S \quad \text{for } n \gg 0.$$

Assume $p = (0,0)$, s.t. $m^n = I^n \mathcal{O}$ where $I = (x, y) \subset k[x, y]$.

$$\begin{array}{c} k[x, y] \\ \swarrow (I, I^n) \\ \text{(Artinian)} \end{array} \rightarrow \mathcal{O}_{p, k[x, y]} / (I, I^n) = \mathcal{O}_{p, F} / m_{p, F}^n = \mathcal{O} / m^n.$$

Intersection Theory.

Definition: Let F and G be the two plane curves, we say F and G intersect properly at p if F and G has no common component passing p .

($F = xY$, $G = x(x^2 + Y^3)$ not intersect properly at $p = (0,0)$).

We say F and G intersect transversally at p if p is a single point both of F, G , and a tangent line to F at p is different from that of G .

Theorem: There is a unique intersection number $I(p, F \cap G)$ for all plane curves F, G and any point $p \in A_k^2$ satisfying:

1) $I(p, F \cap G) \in \mathbb{N}$, if F and G intersects properly at p .

2) $I(p, F \cap G) = 0$, iff $p \notin F \cap G$, $I(p, F \cap G)$ only depends on the component of F and G through p .

3) $I(p, F \cap G)$ is invariant under the affine change of coordinate.

4) $I(p, F \cap G) = I(p, G \cap F)$.

5) $I(p, F \cap G) \geq m_p(F) \cdot m_p(G)$, " $=$ " holds iff they have no common line. tangent.

b): Additivity: if $\bar{F} = \prod_i \bar{F}_i^{r_i}$, $G = \prod_j G_j^{s_j}$, then:

$$I(p, \bar{F} \cap G) = \sum_{i,j} r_i s_j I(p, \bar{F}_i \cap G_j).$$

7): $I(p, \bar{F} \cap G) = I(p, \bar{F} \cap (G + \bar{F}H))$ for any $H \in k[x_1, \dots, x_n]$.

Actually: $I(p, \bar{F} \cap G) = \dim_k \mathcal{O}_{p/A^2} / (\bar{F}, G).$

Projective Algebraic Set.

k -algebraically closed set.

Definition : Define the n -space for k denote by the IP_k^n , or IP^n , to be the set of the equivalence classes of points in $A_k^{n+1} \setminus \{0, \dots, 0\}$, under the equivalent relation " $\sim$ ";

$$(a_0, a_1, \dots, a_n) \sim (b_0, b_1, \dots, b_n) \Leftrightarrow \exists \lambda \neq 0 \in k, \text{ s.t. } b_i = \lambda a_i, \text{ for } i = 0, 1, \dots, n.$$

$$\pi: A_k^{n+1} \setminus \{0, \dots, 0\} \longrightarrow IP_k^n, (a_0, \dots, a_n) \mapsto [a_0, \dots, a_n].$$

Let $S = k[x_0, \dots, x_n]$, $p = (a_0, \dots, a_n) \in IP_k^n$, $f \in S$.

$$S = \bigoplus_{d=0}^{\infty} S_d$$

$$S_d = \left\{ \sum_{|\alpha|=d} c_\alpha x^\alpha \mid c_\alpha \in k \right\}, \alpha = (a_0, \dots, a_n), x^\alpha = x_0^{a_0} \dots x_n^{a_n}, |\alpha| = \sum |\alpha_i|.$$

$$f \in S_d, f(\lambda a_0, \dots, \lambda a_n) = \lambda^d f(a_0, \dots, a_n).$$

For any homogenous polynomial $f \in S$, $p \in IP_k^n$, define

$$f(p) = \begin{cases} 0 & , \text{ if } f(a_0, \dots, a_n) = 0 \\ 1 & , \text{ otherwise.} \end{cases}$$

A graded ring is ring S with a decomposition: $S = \bigoplus_{d=0}^{\infty} S_d$ of S into a direct sum of subgroups S_d , such that $S_d + S_e \subseteq S_{d+e}$ of all d, e , elements of S_d is called homogenous of degree d , then S_0 is a subring of S and S_d is an S_0 -module.

Example : $S = k[x_0, \dots, x_n]$, $S_0 = k$,

An ideal of S is called homogenous if it can be generated by homogenous elements,

Lemma : Let S be a graded ring, and $\mathcal{U}$ an ideal of S : iff:

1) $\mathcal{U}$ is a homogenous ideal of S .

$$2) : \mathcal{U} = \bigoplus_{d=0}^{\infty} (\mathcal{U} \cap S_d)$$

if $\mathcal{U}$ is an ideal of S , so is $\sqrt{\mathcal{U}}$.

Definition: 1) A subset $Y \subseteq \mathbb{P}_k^n$ is called an (projective) algebraic set, if there exists a set, T , of homogenous polynomial in $S = k[x_0, \dots, x_n]$ such that:

$$Y = Z_+(T) = \left\{ p \in \mathbb{P}_k^n \mid f(p) = 0, \forall f \in T \right\}.$$

2) For any homogenous ideal $\mathcal{I}$ of S , define $Z_+(\mathcal{I}) = Z_+(T)$ where T is the set of all homogenous ideals in $\mathcal{I}$.

3). For any subset Y of $\mathbb{P}_k^n$, define $I(Y)$ to be the ideal generated by $\{f \in S \mid f \text{ is hom and } f(p) = 0, \forall p \in Y\}$.

4). For any algebraic set Y of $\mathbb{P}_k^n$, define the homogenous coordinate ring of Y to be $S(Y) = S/I(Y)$ (graded ring).

Example: If $Y = \{p\} = \{(a_0, \dots, a_n)\}$ then $I(Y) = (a_i x_j - a_j x_i \mid 0 \leq i < j \leq n)$.

view $I(Y)$ as an ideal of the affine coordinate ring, $k[x_0, \dots, x_n]$ of $\mathbb{A}_k^{n+1}$. we have

$Z(I(Y))$ is the line passing through p and the origin point of $\mathbb{A}_k^{n+1}$. $I(Y)$ is a prime ideal of S but not a maximal ideal,

Proposition: 1) $\emptyset, \mathbb{P}_k^n$ are algebraic set,

2): the union of two algebraic sets and the intersection of a finitely many algebraic sets are algebraic set in $\mathbb{P}_k^n$.

Definition: Define a Zariski's topology on $\mathbb{P}_k^n$, by taking open subset to be the complement of a algebraic set of $\mathbb{P}_k^n$.

For any projective algebraic subset Y of $\mathbb{P}_k^n$, define homogenous coordinate ring to be:

$$S(Y) = S/I(Y).$$

$I(Y)$ is the homogenous ideal of S and hence $S(Y)$ is a graded ring.

Example:

a): $p = (a_0, \dots, a_n)$ is a projective algebraic set.

$$I(p) = (a_i x_j - a_j x_i \mid 0 \leq i < j \leq n)$$

b): $H_i = Z_+(x_i) = \{p = (a_0, \dots, a_n) \mid a_i = 0\}$.

A quasi-projective subset of $\mathbb{P}_k^n$ is an open subset of a projective subset of $\mathbb{P}_k^n$.

c): Let $U_i = \mathbb{P}_k^n - H_i = \{p = (a_0, \dots, a_n) \mid a_i \neq 0\}$.

Define $\varphi_i: U_i \rightarrow \mathbb{A}_k^n$, by: $\varphi_i(a_0, \dots, a_n) = (\frac{a_0}{a_i}, \dots, \frac{a_n}{a_i})$, then φ_i is bijective.

Theorem: φ_i is a homeomorphism.

Proof: It suffices to show that the closed subset of U_i and $\mathbb{A}_k^n$ are 1-1 correspondent.

via φ_i , say $i=0$,

$$\varphi_0: U_0 \rightarrow \mathbb{A}_k^n. \quad \varphi_0(p) = (\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0})$$

A subset of U_0 has the form: $Z_+(T) \cap U_0$ for some $T \in S^h$.

Proposition: Homogenous Nullstellensatz.

1) For any homogenous ideal $\mathfrak{a}$ of $S = k[x_0, \dots, x_n]$.

$$I(Z_+(\mathfrak{a})) = \sqrt{\mathfrak{a}}$$

2). For any subset Y of $\mathbb{P}^n$:

$$Z_+(I(Y)) = \overline{Y}.$$

3). The map $Z_+ \mapsto Z_+(Z_+)$ and $Y \mapsto I(Y)$ define a bijection between the set of radical homogenous ideals of S and closed subset of $\mathbb{P}^n$.
(not equals to $S_+ = (x_0, \dots, x_n)$).

4). Under the above bijection, we have

$$\left\{ \begin{array}{l} \text{irreducible closed subsets} \\ \text{of } \mathbb{P}_k^n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{homogenous prime ideal} \\ \text{of } S \text{ not equals to } S_+ \end{array} \right\}.$$

$$\left\{ \begin{array}{l} \text{points of } \mathbb{P}_k^n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Maximal element of the set} \\ \text{of radical homogenous ideals of } S \\ \neq S_+ \end{array} \right\}.$$

5). $\mathbb{P}_k^n$ is a Noetherian topological space.

Let Y be a closed subset of $\mathbb{P}^n$,

$$S(Y) = S/I(Y).$$

For any $g \in S(Y)$, take the $\tilde{g} \in S$, define

$$D_+(g) = \{p \in Y \mid \tilde{g}(p) \neq 0\}$$

$D_+(g)$ is independent of the choice of $\tilde{g}$. $\{D_+(g) \mid g \in S(Y)\}$ form a topological basis of Y .

Proposition: Let Y be a closed subset of $\mathbb{P}^n$, $S(Y) = S/I(Y)$, we have 1-1 correspondence,

$$\begin{aligned} \left\{ \begin{array}{l} \text{closed subset of} \\ Y \end{array} \right\} &\longleftrightarrow \left\{ \begin{array}{l} \text{homogeneous radical ideals of} \\ S(Y) \text{ not equal to } S_+(Y) = S_+/I(Y) \end{array} \right\} \\ \left\{ \begin{array}{l} \text{irreducible closed subsets} \\ \text{of } Y \end{array} \right\} &\longleftrightarrow \left\{ \begin{array}{l} \text{homogeneous prime ideals of } S(Y) \\ \text{not equal to } S_+(Y) \end{array} \right\} \\ \left\{ \begin{array}{l} \text{points of } Y \end{array} \right\} &\longleftrightarrow \left\{ \begin{array}{l} \text{maximal elements of the set of} \\ \text{homogeneous radical ideals of } S(Y) \\ \neq S_+(Y) \end{array} \right\}. \end{aligned}$$

Definition: A quasi-affine variety in $\mathbb{A}^n$ is an open subset of an irreducible closed subset of $\mathbb{A}^n$.

A (quasi) affine projective variety in $\mathbb{P}^n$ is (an open subset) of an irreducible closed subset of $\mathbb{P}^n$.

Let X be a quasi-affine variety of $\mathbb{A}^n$, A function $f: X \rightarrow k$ is called regular at $p \in X$ if there exist an open nbhd U of p in X , and $g, h \in k[x_0, \dots, x_n]$, such that h nowhere vanishing on U , and $f = g/h$ on U .

Let X be a quasi-projective variety over $\mathbb{P}^n$, a function $f: X \rightarrow k$ is called regular at $p \in X$, if there exists an open nbhd U of p in X , and two homogeneous polynomials $g, h \in S = k[x_0, \dots, x_n]$ of same degree, such that h nowhere vanishing on U , and $f = g/h$ on U , if f is regular on each point of X , we call X is a regular function on Y .

Corollary: $f: X \rightarrow k$ a regular function, then f is continuous and satisfies the uniqueness property.

A variety over k is a quasi-affine variety over k or a quasi-projective variety over k .

Let X, Y be two varieties, A morphism $\varphi: X \rightarrow Y$ is a continuous map st for any open subset V of Y and any regular function $f: V \rightarrow k$,

$$f \circ \varphi|_{\varphi^{-1}(V)} : \varphi^{-1}(V) \rightarrow k \text{ is regular.}$$

Two varieties are called isomorphic, if there are two regular $\varphi: X \rightarrow Y, \psi: Y \rightarrow X$, st: $\varphi \circ \psi = \psi \circ \varphi = \text{id}$

An isomorphism between varieties is a homeomorphism, vice not-versa.

Definition: Let X be a variety, Denote by $\mathcal{O}(X)$ the ring of regular functions on X ,
Let $p \in X$ define the local ring $\mathcal{O}_p(X)$ of p in X to be the ring of germ of regular function on X near p . In other words,

$$\mathcal{O}_p(X) = \varinjlim_{p \in U} \mathcal{O}(U)$$

An element of $\mathcal{O}_p(X)$ is an equivalence classes of pairs (U, f) where U is an open nbhd of p in X and $f \in \mathcal{O}(U)$, two such pairs equivalent $(U, f) \sim (V, g)$ iff $f = g$ on $U \cap V$.

$\mathcal{O}_p(X)$ is a local ring with residue field k .

Define the function field $K(X)$ over X to be:

$$K(X) = \varinjlim_U \mathcal{O}(U).$$

Thm: Let X be an affine variety of $\mathbb{A}^n$ with affine coordinate ring:

$$A(X) = A / I(X).$$

where $A = k[x_1, \dots, x_n]$, then:

$$1) : \mathcal{O}(X) \cong A(X)$$

2) : $\forall p \in X$, let $\mathfrak{m}_{p,X}$ be the maximal ideal of $A(X)$ consisting of functions vanishing

at p , then $\hat{\mathcal{O}}_p(x) \cong A(x)_{m_p(x)}$

3). $K(x)$ is the fraction field of $A(x)$.

Proposition 1: The map: $\varphi_i: U_i \rightarrow A^n$ given by $\varphi_i(a_0, \dots, a_n) = (\frac{a_0}{a_i}, \dots, \frac{\hat{a_i}}{a_i}, \dots, \frac{a_n}{a_i})$

A quasi-affine or quasi-projective varieties are called varieties,

Let X be a variety, $p \in X$.

$\mathcal{O}(X)$, the ring of local regular functions on X .

$\mathcal{O}_p(X)$, the ring of germs of regular functions at X near p .

$K(X)$, the function field of X .

$$\mathcal{O}_p(X) = \varinjlim_{p \in U \subset X} \mathcal{O}(U), \quad K(X) = \varinjlim_{\emptyset \neq U \subset X} \mathcal{O}(U).$$

Definition: Let S be a graded ring, T is a multiplicative system of S consisting of homogeneous element.

is a isomorphism. Then $T^{-1}S$ has a natural graded structure such that the canonical homomorphism $S \rightarrow T^{-1}S$ preserve the grading. then $(T^{-1}S)_0$ is a subring of $T^{-1}S$.

$$(T^{-1}S)_0 = \left\{ \frac{a}{t} \mid a \in S^h, t \in T, \deg a = \deg t \right\}.$$

Example: Take $S = k[x_0, \dots, x_n]$ $g=0$, $T = S^h \setminus \{0\}$,

$$\begin{aligned} (T^{-1}S)_0 &= \left\{ \frac{f}{g} \mid f, g \in S^h, g \neq 0, \deg(f) = \deg(g) \right\} \\ &= k\left(\frac{x_0}{x_1}, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right). \end{aligned}$$

1) Let p be a prime homogeneous ideal of S . Take $S \setminus p = T$. Define $(T^{-1}S)_0$ by $S_{(p)}$. Then $(S)_{(p)}$ is a local ring with maximal ideal.

$$\begin{array}{ccccc} S_{(p)} & = & (T^{-1}S)_0 & \subset & T^{-1}S \hookleftarrow S \\ & & \cup & & \cup & & \cup \\ & & pT^{-1}S \cap S_{(p)} & & pT^{-1}S & & p \end{array}$$

Any element of $S_{(p)}$ has the form $\frac{a}{t}$ for some $a \in S^h$, $t \in S^h \setminus p$ with $\deg(a) = \deg(t)$

If $a \in \mathcal{P}$ let $\frac{a}{t} \in \mathcal{P}T^1S \cap S_{(\mathcal{P})}$, so if $\frac{a}{t} \notin \mathcal{P}T^1S \cap S_{(\mathcal{P})}$, let $a \in \mathcal{P}$, then $\frac{a}{t} \in (S_{(\mathcal{P})})^*$ with inverse $\frac{t}{a}$.

If $\mathcal{P}T^1S = T^1S$ then $1 = \sum_i a_i \frac{b_i}{t_i}$ for some $a_i \in \mathcal{P}$, $b_i \in S$, $t_i \in T$. so, $\frac{1}{1} = \frac{b}{s}$ for some $b \in \mathcal{P}$, $s \in T$, that is $s'(s-b) = 0$ for some $s' \in T$, $T \ni s'S = s'b \in \mathcal{P}$, contradiction. So $\mathcal{P}T^1S \neq T^1S$ and $\mathcal{P}(T^1S) \cap S_{(\mathcal{P})} \neq S_{(\mathcal{P})}$. This proves $S_{(\mathcal{P})}$ is a local ring with maximal ideal $\mathcal{P}T^1S \cap S_{(\mathcal{P})}$.

2) Let $f \in S^h$, Take $T = \{1, f, f^2, \dots\}$. Denote $(S_{\mathcal{P}})_0$ by $S(f)$. Any element of $S(f)$ has the form a/f^n for some $a \in S^h$, $n \in \mathbb{N}$, st: $\deg(a) = n \deg(f)$.

Example: Take $S = k[x_0, \dots, x_n]$, $f = x_0$, Then any element of $S_{(x_0)}$ has the form

$$\frac{f(x_0, \dots, x_n)}{x_0^m} = f(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}) \in k[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}], \text{ we have}$$

$$S_{(x_0)} \cong k[\frac{x_1}{x_0}, \frac{x_2}{x_0}, \dots, \frac{x_n}{x_0}].$$

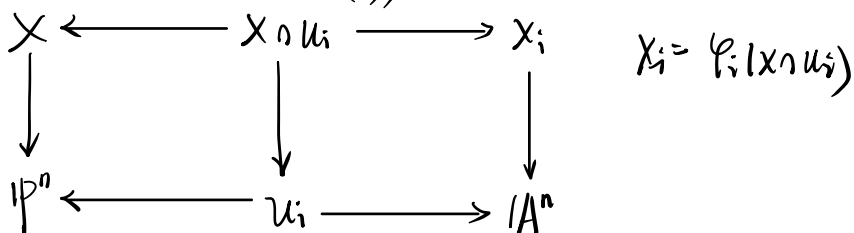
Theorem: Let $X \subseteq \mathbb{P}^n$ be a projective variety with homogeneous coordinate ring $S(X)$. Then

1). $\mathcal{O}(X) \cong k$.

2). For any $p \in X$, let $\mathfrak{m}_p \subseteq S(X)$ be the homogeneous ideal generated by the set of homogeneous elements $f \in S(X)$ s.t. $f(p) = 0$, then:

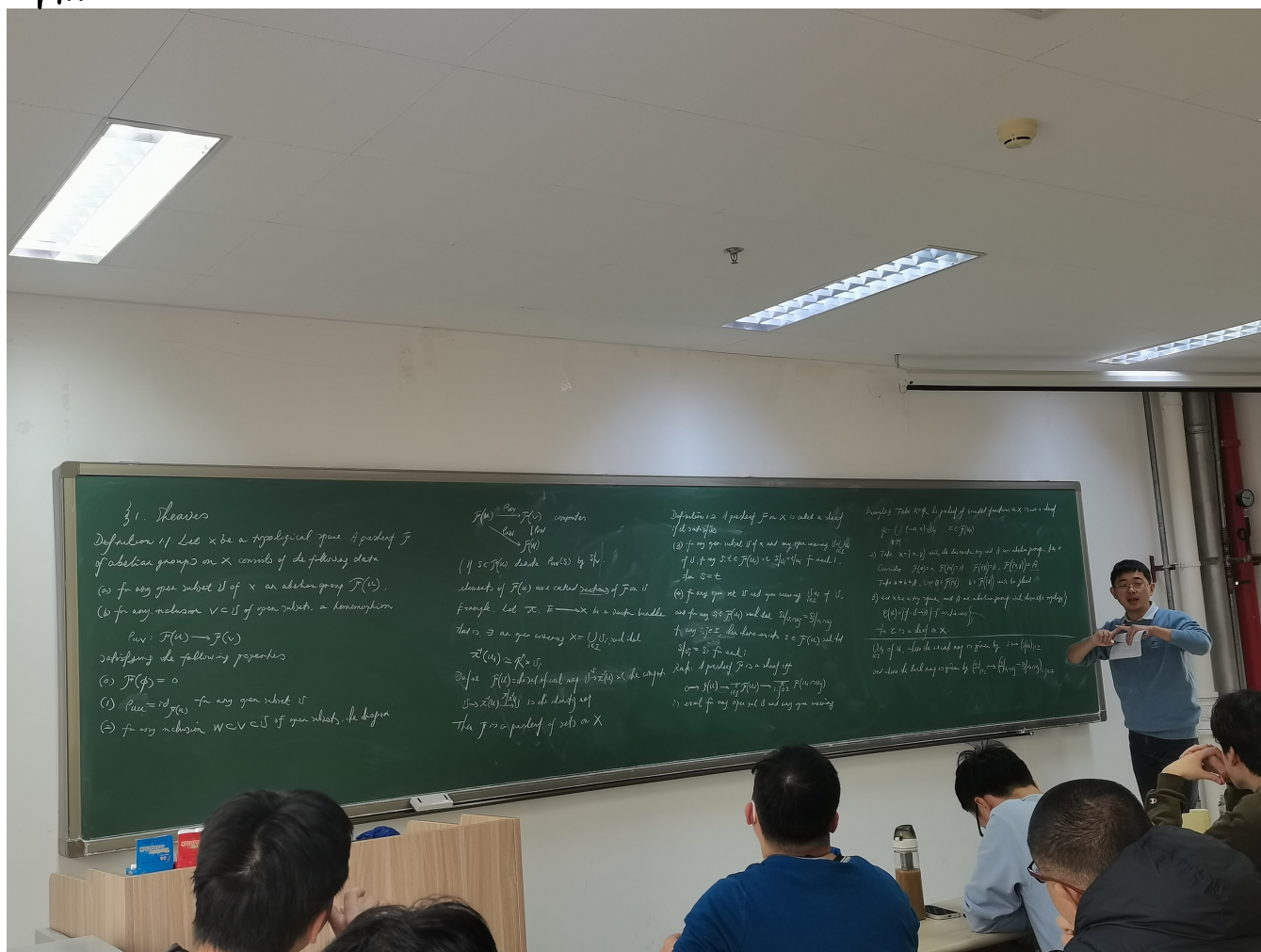
$$\mathcal{O}_p(X) \cong (S(X))_{(\mathfrak{m}_p)}$$

3). $K(X) \cong S(X)_{(0)}$



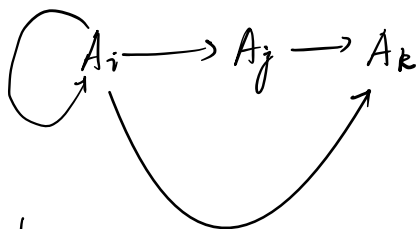
Part II. Schemes.

This is a section about sheaves.



Definition 1.1 A direct set is partially ordered set $(I, \leq)$, such that for any $i, j \in I$, there exists $k \in I$, s.t. $i \leq k, j \leq k$.

A direct system $(A_i, \phi_{ij})_{i, j \in I}$ consists of a set A_i for each $i \in I$, and a map $\phi_{ij}: A_i \rightarrow A_j$, for each $i \leq j$, satisfying: $\phi_{ij} \circ \phi_{ki} = \phi_{jk}$, $\phi_{ii} = \text{id}_{A_i}$.

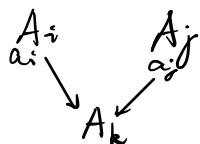


A direct limit of a direct system is:

$$\coprod_{i \in I} A_i / \sim := \varinjlim_{i \in I} A_i$$

where $a_i \in A_i, a_j \in A_j, a_i \sim a_j \iff \exists k \in I, \text{ st. } i \leq k, j \leq k, \phi_{ik}(a_i) = \phi_{jk}(a_j)$.

For each $i \in I$, we get a canonical map $A_i \xrightarrow{f_i} \varinjlim_{j \in I} A_j$
 Such that $f_i = f_j \circ \phi_{ij}$, any $i \leq j$. (The universal property).



Definition: Let X be a topo. space and $x \in X$. For any open nbhd V of x in X , we say $U \subseteq V$, if $V \subseteq U$. Then the set of open nbhd of x in X is a direct set. Let $\mathcal{F}$ be a presheaf on X . Then $\{\mathcal{F}(U) \mid U \text{ nbhd of } x\}$ is a direct system when $U \subseteq V$ runs over all nbhds of x in X . Define

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U)$$

the germ of sheaf $\mathcal{F}$ at x .

Definition: Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves on X , a homomorphism $\phi: \mathcal{F} \rightarrow \mathcal{G}$ is a family of homomorphism between Abelian groups: $\phi(U): \mathcal{F}(U) \rightarrow \mathcal{G}(U)$, st for any including map:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \rho_{UV} \downarrow & & \downarrow \rho_{UV} \\ \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{G}(V) \end{array} \quad \text{commutes.}$$

For any $x \in X$, $\phi: \mathcal{F} \rightarrow \mathcal{G}$ induced a homomorphism between $\phi_x: \mathcal{F}_x \rightarrow \mathcal{G}_x$.
 So we get a Category of presheaves on X . (An Abelian category)

Proposition: Let $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism between sheaves on X , then ϕ is an isomorphism, iff $\phi_x: \mathcal{F}_x \rightarrow \mathcal{G}_x$ is an isomorphism.

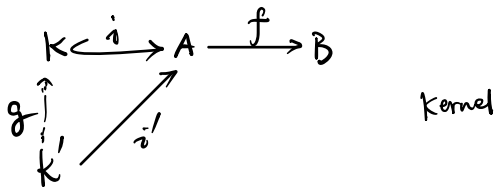
Today's course is about the **Abelian Category**.

$\mathcal{C}$ is an additive category, if $\forall A, B, C \in \text{Ob}(\mathcal{C})$, $A \times B$ exists, $\text{Hom}(A, B)$ is ^{an} Abelian group, and:

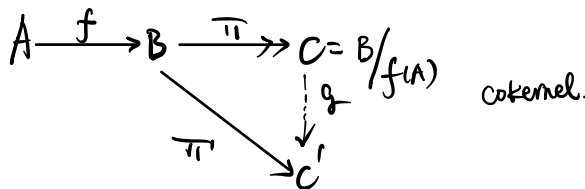
$$\text{Hom}(A, B) \times \text{Hom}(B, C) \longrightarrow \text{Hom}(A, C)$$

is an homomorphism of Abelian groups.

Ex: in Additive Category, $A \times B \cong A \oplus B$.

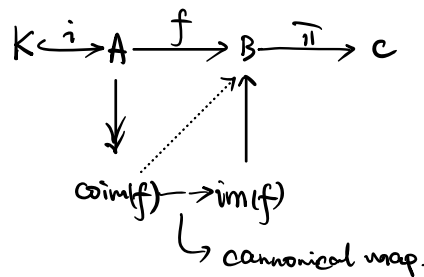


A monomorphism $i: K \rightarrow A$ is called the kernel of $f: A \rightarrow B$ if the composite map $K \xrightarrow{i} A \xrightarrow{f} B$ is zero map, and for any $i': K' \rightarrow A$, there exists a unique $g: K' \rightarrow K$ such that the left diagram commutes



An epimorphism $\pi: B \rightarrow B/f(A)$ is called the cokernel of $f: A \rightarrow B$ if $A \xrightarrow{f} B \xrightarrow{\pi} B/f(A)$ is zero, and for any $\pi': B \rightarrow C'$ with $\pi' \circ f = 0$, $\exists!$ $g: C \rightarrow C'$ s.t. the left diagram commutes.

$$\begin{aligned} \text{im } f &= \text{Ker}(\text{coker } f) \\ \text{coim } f &= \text{coker}(\text{ker } f) \\ &= A/\text{ker } f \end{aligned}$$



Let $\mathcal{C}$ be an Addition Category, A zero object 0 in $\mathcal{C}$ is an object s.t. $\text{Hom}(0, 0) = 0$. The 0 objects are isomorphism.

An Abelian Category $\mathcal{C}$ is an addition category, with 0 object, for any $f: A \rightarrow B$ in $\mathcal{C}$, the kernel and cokernel of f exist, and the canonical map of $\text{coim}(f) \rightarrow \text{im}(f)$ is an isomorphism.

Ex: in Abelian category: bijective $\Leftrightarrow$ isomorphism.

Suppose: $u: A \rightarrow B$ is a monomorphism in an Abelian category, we say A is a subobject of B , denoted by $A \subset B$.

Let C be the cokernel of u , we say C is the quotient of B by A , denoted by B/A .

In an Abelian Category, a sequence of morphism $A \xrightarrow{u} B \xrightarrow{v} C$ is called exact if $v \circ u = 0$, and the canonical morphism $\text{coim}(u) \xrightarrow{\cong} \text{ker}(v)$ is an isomorphism.

A short exact form is an exact sequence of the five:

$$0 \rightarrow A \xrightarrow{\quad} B \xrightarrow{\quad} C \rightarrow 0$$

$\downarrow$ injective $\downarrow$ epimorphism

Let $F: \mathcal{E} \rightarrow \mathcal{D}$ be a covariant functor between Abelian categories. We say F is an additive functor if for any A, B in $\text{Ob}(\mathcal{E})$, the map:

$$\begin{array}{c}
 \mathcal{E}: \quad A \xrightarrow{u} B \xrightarrow{v} C \\
 \quad \quad \quad \searrow \quad \nearrow \\
 \quad \quad \quad \quad uv \\
 \downarrow F \\
 \mathcal{D}: \quad F(A) \xrightarrow{F(u)} F(B) \xrightarrow{F(v)} F(C) \\
 \quad \quad \quad \searrow \quad \nearrow \\
 \quad \quad \quad \quad F(uv)
 \end{array}$$

$F: \text{Hom}_{\mathcal{E}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B))$
 is homomorphism of Abelian groups.

Ex. If f is additive, then $F(A \oplus B) = F(A) \oplus F(B)$.

Proposition: Let X be a topological space, then the category of presheaves on X is an Abelian category. Let $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of presheaves on X , then the kernel, cokernel, coim , im of ϕ are the presheaves defined by.

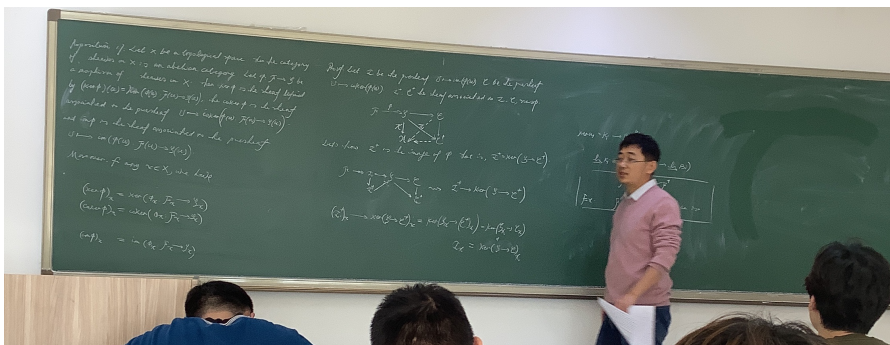
$$\begin{aligned}
 (\ker \phi)(U) &= \ker(\phi_U: \mathcal{F}_U \rightarrow \mathcal{G}_U) \\
 (\text{coker } \phi)(U) &= \text{coker}(\phi_U: \mathcal{F}_U \rightarrow \mathcal{G}_U) \\
 (\text{coim } \phi)(U) &= \text{coim}(\phi_U: \mathcal{F}_U \rightarrow \mathcal{G}_U) \\
 (\text{im } \phi)(U) &= \text{im}(\phi_U: \mathcal{F}_U \rightarrow \mathcal{G}_U)
 \end{aligned}$$

Proposition: Let $\mathcal{F}$ be a presheaf on X , there exists a pair $(\mathcal{F}^+, \theta)$ consisting of the following: a sheaf $\mathcal{F}^+$, and a morphism $\theta: \mathcal{F} \rightarrow \mathcal{F}^+$ of presheaves, satisfying the following property: for any $\mathcal{G}$ and any morphism $\psi: \mathcal{F} \rightarrow \mathcal{G}$

$\nwarrow$ universal

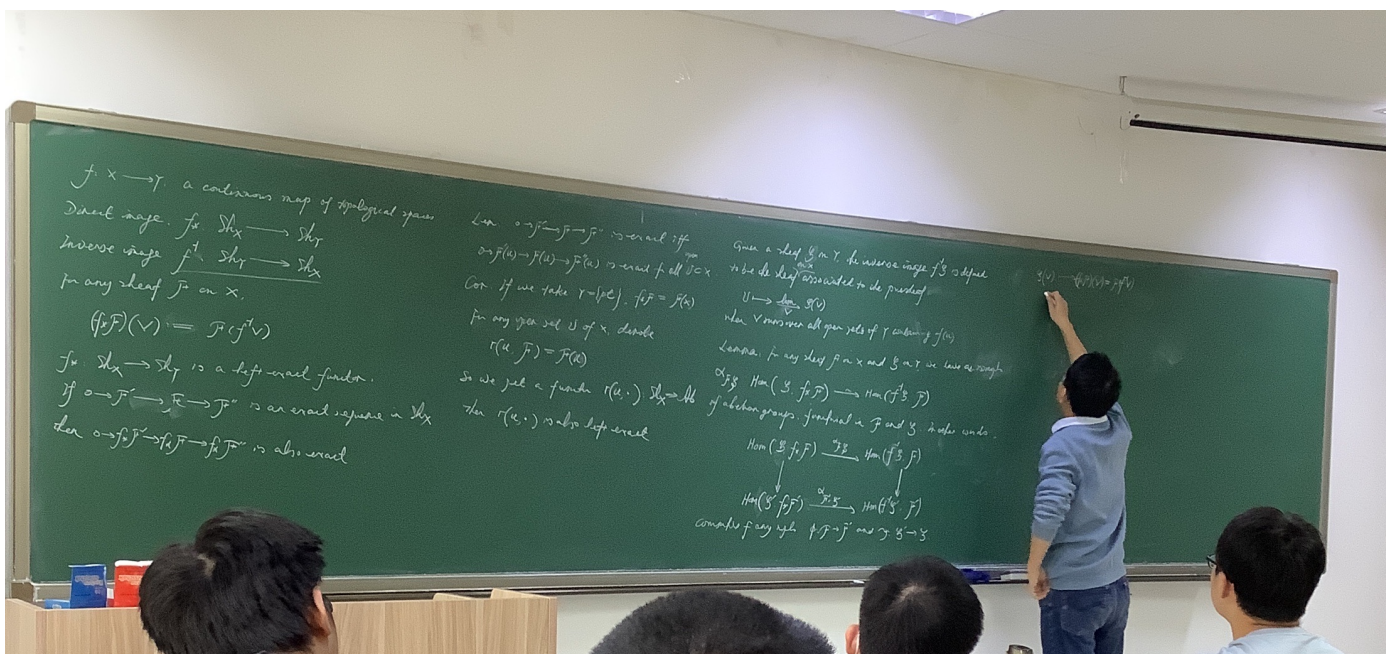
$$\begin{array}{ccc}
 \mathcal{F} & \xrightarrow{\theta} & \mathcal{F}^+ \\
 \searrow \psi & & \swarrow \psi^+ \\
 & \mathcal{G} &
 \end{array}$$

there exists a unique morphism $\varphi^*: \mathcal{F}_x^* \rightarrow \mathcal{G}_x$ st. $\mathcal{G}^* \circ \varphi = \varphi$.



Definition: Let $f: X \rightarrow Y$ be a continuous map of topological spaces. Let $\mathcal{F}_x$ be a sheaf on X . Define the direct image of $\mathcal{F}_x$ on Y to be the sheaf $(f_* \mathcal{F}_x)(V) = \mathcal{F}_x(f^{-1}V)$ for any open set V of Y .

$\nabla > W$



Let $\mathcal{C}, \mathcal{D}$ be 2 categories, and let $U_*: \mathcal{C} \rightarrow \mathcal{D}$ and $U^*: \mathcal{D} \rightarrow \mathcal{C}$ be 2 functors. we say (U^*, U_*) is an adjoint pair if we have a bijection:

$$\text{Hom}(U^*(Y), X) \xrightarrow{\cong} \text{Hom}(Y, U_*(X)).$$

for any objects: X in $\mathcal{C}$ and Y in $\mathcal{D}$. which are functorial in X and Y .

Example: $\text{Sh}_X \xrightarrow{i} \text{Psh}_X$ (a forgettable functor)

$$\text{Psh}_X \xrightarrow{+} \text{Sh}_X \quad (\text{sheafification})$$

Let $\mathcal{F}$ be a presheaf on X , $\mathcal{G}$ be a sheaf on X .

$$\text{Hom}_{\text{Psh}_X}(\mathcal{F}, \mathcal{G}) \xrightarrow{\cong} \text{Hom}_{\text{Sh}_X}(\mathcal{F}^t, \mathcal{G})$$

$(+ , i)$ is an adjoint pair.

Taking $Y = U_*(X)$, the identity in $\text{Hom}(U_*(X), U_*(X))$, and we get a morphism:

$$U^t U_*(X) \longrightarrow X \text{ in } \mathcal{C}.$$

and hence a transformation $U^t U_* \xrightarrow{\quad} \text{Id}: \mathcal{C} \rightarrow \mathcal{C}.$

Taking $X = U^t(Y)$, we get a morphism, $Y \rightarrow U_* U^t(Y)$, and hence a transformation between functors, $\text{Id} \rightarrow U_* U^t: \mathcal{D} \rightarrow \mathcal{D}.$

So for any sheaf $\mathcal{F}$ on X , we get a morphism:

$$f^t f_* \mathcal{F} \longrightarrow \mathcal{F}.$$

and for any sheaf $\mathcal{G}$ on Y and hence a morphism:

$$\mathcal{G} \longrightarrow f_* f^t \mathcal{G}.$$

$$\alpha_{\mathcal{F}, \mathcal{G}}: \text{Hom}(f^t \mathcal{G}, \mathcal{F}) \longrightarrow \text{Hom}(\mathcal{G}, f_* \mathcal{F})$$

$$f^t \mathcal{G} \xrightarrow{\alpha} \mathcal{F} \hookrightarrow \left(\mathcal{G} \longrightarrow f_* f^t \mathcal{G} \xrightarrow{f_* (\alpha)} f_* \mathcal{F} \right).$$

$$\left(f^t \mathcal{G} \xrightarrow{f^t \beta} f^t f_* \mathcal{F} \rightarrow \mathcal{F} \right) \rightsquigarrow \mathcal{G} \xrightarrow{\beta} f_* \mathcal{F}.$$

Lemma:

Let $f: X \rightarrow Y$, $g: Y \rightarrow Z$ be two continuous maps of topological space, we have canonical morphisms between functors:

$$g_* \circ f_* \xrightarrow{\cong} (g \circ f)_*, \quad \text{sh}_X \longrightarrow \text{sh}_Z.$$

and:

$$f^t \circ g^t \xrightarrow{\cong} (g \circ f)^t, \quad \text{sh}_Z \longrightarrow \text{sh}_X.$$

More precisely, for any sheaf $\mathcal{F}$ on X and $\mathcal{H}$ on Z we have:

$$g_* (f_* \mathcal{F}) \xrightarrow{\cong} (g \circ f)_* \mathcal{F} \quad \text{and}$$

$$f^{-1}(g^*(x)) \xrightarrow{\cong} (g \circ f)^*(x).$$

Corollary: Let $f: X \rightarrow Y$ be a continuous map, $x \in X$. Let:

$i: \{pt\} \rightarrow X$
be the map with image x .

$$\begin{array}{ccc} \{pt\} & \xrightarrow{i} & X \\ & \searrow f \circ i & \swarrow f \\ & & Y \end{array}$$

Let $\mathcal{G}$ be a sheaf on Y , we have,

$$i^*(f^*\mathcal{G}) \cong (f \circ i)^*\mathcal{G}$$

$$(f^*\mathcal{G})_x \cong \mathcal{G}_{f(x)}.$$

So, $f^*: \text{Sh}_Y \rightarrow \text{Sh}_X$ is exact functor.

So, here, we end the chapter
of Sheaves!

§. Schemes

For any ring A , let $\text{Spec} A$ be the set of all prime ideals of A , for any ideal $\mathfrak{a}$ of A , let $V(\mathfrak{a}) = \{ \mathfrak{p} \in \text{Spec} A \mid \mathfrak{p} \supseteq \mathfrak{a} \}$.

Example: $A = k[x_1, \dots, x_n]$,

Proposition: (1): $V(0) = \text{Spec} A$, $V(A) = \emptyset$,

$$(2): V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cdot \mathfrak{b})$$

$$(3): \bigcap_{i \in I} V(\mathfrak{a}_i) = V\left(\sum_{i \in I} \mathfrak{a}_i\right)$$

$$(4): V(\mathfrak{a}) \subseteq V(\mathfrak{b}) \iff \mathfrak{b} \subseteq \sqrt{\mathfrak{a}}.$$

We define the Zariski topology on $\text{Spec} A$ by taking open sets to be a complement of $V(\mathfrak{a})$ for all ideals $\mathfrak{a}$ of A .

$$\text{Let } D(f) = \{ \mathfrak{p} \in \text{Spec} A \mid f \notin \mathfrak{p} \}$$

Proposition: For any $f \in A$, $D(f)$ is open and they form a topological basis of $\text{Spec} A$. Moreover $\text{Spec} A$ is quasi-compact. ($D(f)$ is quasi-compact).

$$\begin{array}{c} \curvearrowright \\ f=1. \end{array}$$

We should define a sheaf of rings on $\text{Spec} A$. For any open set U of $\text{Spec} A$, let $\mathcal{O}_{\text{Spec} A}(U)$ be the set of maps:

$$s: U \longrightarrow \coprod_{\mathfrak{p} \in \text{Spec} A} A_{\mathfrak{p}}.$$

Satisfies the following condition:

$$(1) s(\mathfrak{p}) \in A_{\mathfrak{p}}.$$

(2), for any $\mathfrak{p} \in U$, $\exists$ a nbhd $\bigvee$ of $\mathfrak{p}$ in U and $a, t \in A$, such that for all $g \in V$, we have $t \notin \mathfrak{g}$ and $s|_V = \frac{a}{t} \in A_g$.

So we get a sheaf of ring $\mathcal{O}_{\text{Spec} A}$ on $\text{Spec} A$, that is called the structured sheaf on $\text{Spec} A$.

Definition : A ringed space is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X . A morphism from a ringed space $(X, \mathcal{O}_X)$ to the other one $(Y, \mathcal{O}_Y)$ is a pair (f, f^*) where $f: X \rightarrow Y$ is a continuous map and $f^*: \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ is a morphism of sheaves on Y . $f^*: \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X \xrightarrow{\sim} f^! \mathcal{O}_Y \rightarrow \mathcal{O}_X \xrightarrow{\sim} (f^! \mathcal{O}_Y) \xrightarrow{f_*^*} \mathcal{O}_{X, \pi}$
 $f^*: \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X \xrightarrow{\sim} f_*^*: \mathcal{O}_{X, f(x)} \rightarrow \mathcal{O}_{X, \pi}$ for any $x \in X$.

$$\varinjlim_{f \in U} \mathcal{O}_Y(V) \xrightarrow{\parallel} \varinjlim_{V \in f(U)} \mathcal{O}_X(f^! V) \xrightarrow{\parallel} \varinjlim_{x \in U} \mathcal{O}_{X, \pi}.$$

We call the ringed space $(X, \mathcal{O}_X)$ is a local ringed space if $\mathcal{O}_{X, \pi}$ is a local ring for all $x \in X$. A morphism of locally ringed space is a morphism (f, f^*) of ringed spaces such that $f_x^*: \mathcal{O}_{Y, f(x)} \rightarrow \mathcal{O}_{X, \pi}$ is a local homomorphism of local rings.

A morphism $\varphi: (A, m) \rightarrow (B, n)$ of local rings is called a local morphism if $\varphi(m) \subseteq n$, (epimorphism: $\varphi^*(n) = m$).

So, $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ is a ringed space.

↳ the structured sheaf of $\text{Spec } A$.

Proposition : We write $\mathcal{O}_{\text{Spec } A}$ by $\mathcal{O}$ for the simplicity.

(1) For any $p \in \text{Spec } A$, $\mathcal{O}_p \cong A_p$.

(2) For any $f \in A$, $\mathcal{O}(D(f)) = A_f$. In particular: $\mathcal{O}(\text{Spec } A) \cong A$.

Proof: we have a canonical homomorphism

$$\varphi: \mathcal{O}_p \xrightarrow{\varphi} A_p, \quad s \in \mathcal{O}_{\text{Spec } A}(U) \mapsto s(p)$$

Surjectivity: For any $\frac{a}{t} \in A_p$, with $a \in A$, $t \notin p$, let s be the map:

$$s: D(t) \longrightarrow \bigsqcup_{z \in \text{Spec } A} A_z, \quad s(q) = \frac{a}{t} \in A_z.$$

So $s \in \mathcal{O}_{\text{Spec } A}(\mathcal{D}(t))$, and degree of S at p is a preimage of $\frac{a}{t} \in A_p$.

injectivity: Suppose $\varphi(s) = 0$ for some $s \in \mathcal{O}_p$. we may assume that s is rep by $\frac{a}{t}$ on a nbhd V of p , so $\varphi(s) = \frac{s}{t} = 0 \in A_p$. Hence $at' = 0$ for some $t' \notin p$. Then $\frac{a}{t} \Big|_{\mathcal{D}(t') \cap V} = 0$.

$$\gamma: A_p \rightarrow \mathcal{O}_p, \frac{a}{t} \mapsto s \cdot \mathcal{D}(t) \longrightarrow \bigcap_{t \in \text{Spec } A} A_t \quad \text{st. } s \cdot \mathcal{D}(t) = \frac{a}{t} \in A_t.$$

2): For any $\frac{a}{f^n} \in A_f$, define $\varphi(\frac{a}{f^n}) \in \mathcal{O}_{\text{Spec } A}(\mathcal{D}(f))$ by

$$\varphi\left(\frac{a}{f^n}\right)(p) = \frac{a}{f^n} \in A_p.$$

So, we get a canonical homomorphism $\varphi: A_f \longrightarrow \mathcal{O}_{\text{Spec } A}(\mathcal{D}(f))$.

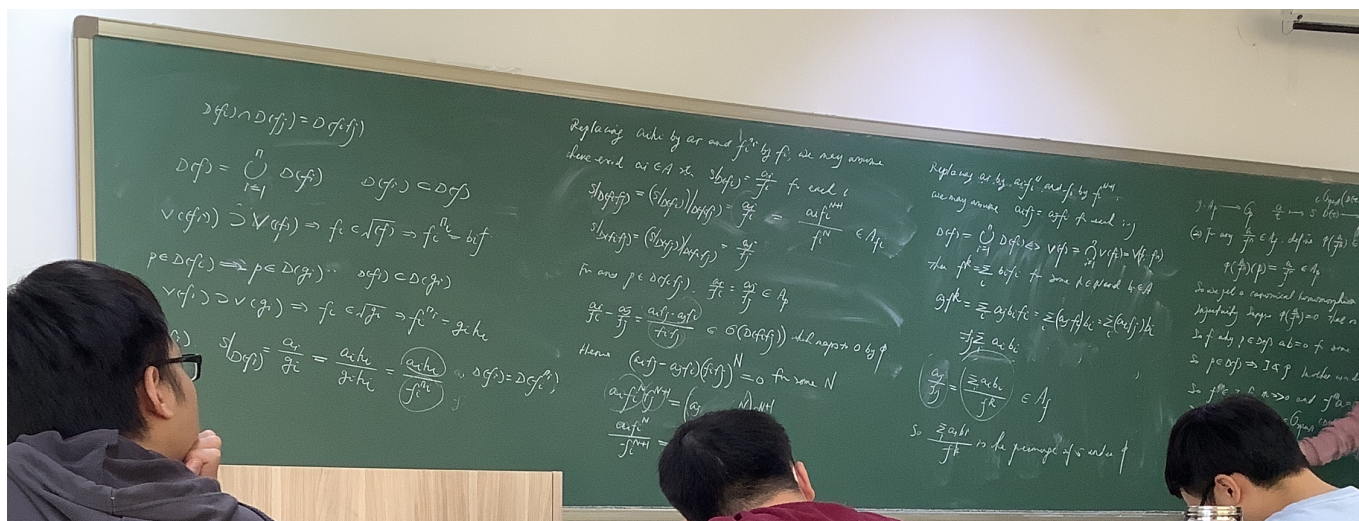
Injectivity: $\varphi(\frac{a}{f^n}) = 0$, that is $\frac{a}{f^n} = 0 \in A_p$, for all $p \in \mathcal{D}(f)$.

So for any $p \in \mathcal{D}(f)$, $a \cdot b = 0$ for some $0 \notin p$. Take $I = \{x \in A \mid xa = 0\}$. So, $p \in \mathcal{D}(f)$,

$\Rightarrow I \not\subseteq p$, vice-versa, $I \subseteq p \Rightarrow f \in p$, so $f^m \in I$, for $m \gg 0$, and $f^m a = 0$

thus $\frac{a}{f^n} = 0 \in A_f$.

Surjectivity:



$(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ is a locally ringed space.

A ringed space which is isomorphic to $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ for some A , is called an affine scheme, a locally space $(X, \mathcal{O}_X)$ is called a scheme if there exists an open covering U_i of X , such that for each i , $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme.

A morphism of schemes is just a morphism of locally ringed space

$$(X, \mathcal{O}_X) \xrightarrow{(f, f^\#)} (Y, \mathcal{O}_Y)$$

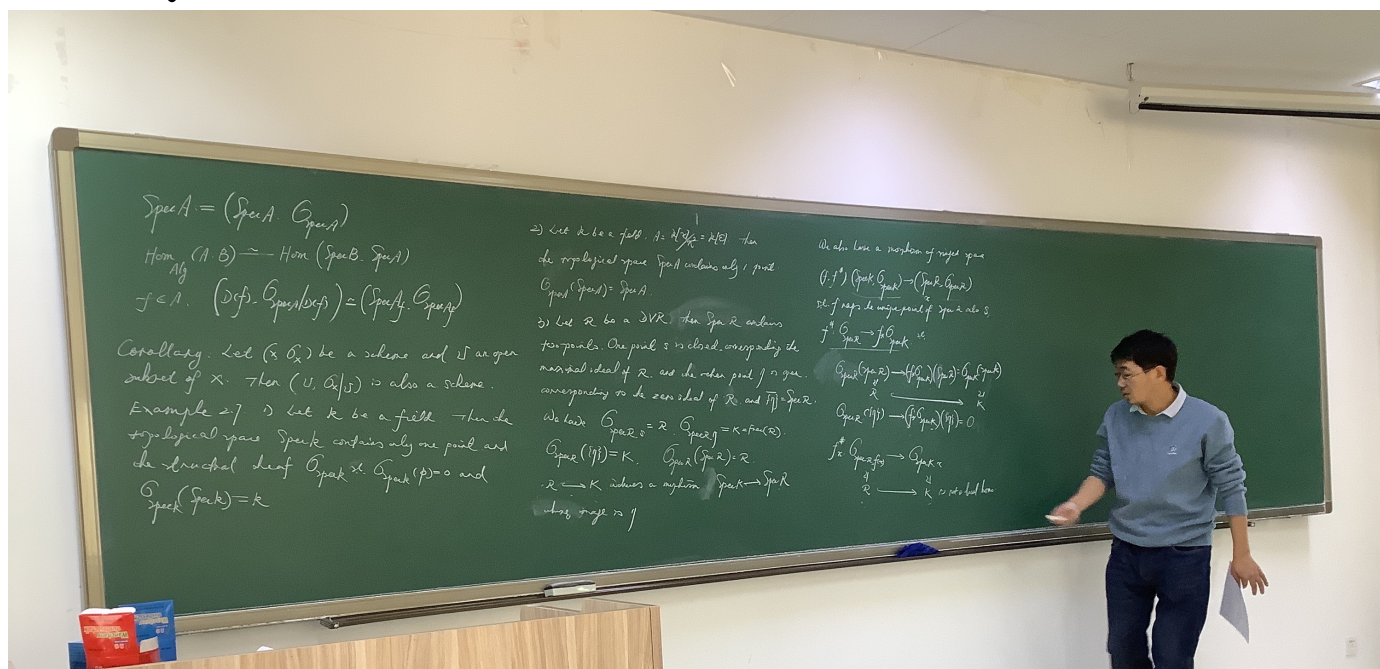
$f: X \rightarrow Y$ continuous map, $f^\#: \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ morphism of sheaves of rings.

Proposition 2.5: (1) If $\phi: A \rightarrow B$ is a homomorphism of rings, then ϕ induces a morphism of locally ringed space

$$(f, f^\#): (\text{Spec } B, \mathcal{O}_{\text{Spec } B}) \rightarrow (\text{Spec } A, \mathcal{O}_{\text{Spec } A})$$

(2). Any morphism of locally ringed spaces $\text{Spec } B \rightarrow \text{Spec } A$ is obtained by this way.

Corollary: if $(X, \mathcal{O}_X)$ is a scheme, then $U \subset X$, $(U, \mathcal{O}_X|_U)$ is also a scheme.



4): Let k be a field, define the affine line: A_k^1 to be $\text{Spec } k[x]$.

It has point η corresponding to the zero ideal, whose closure is the whole space.

The other points, corresponding to maximal ideals of $k[x]$, are closed points,

$$\{\text{closed points of } A_k^1\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{irreducible polynomial} \\ \text{in } k[x] \end{array} \right\}.$$

In particular, if k is algebraically closed,

$$\{\text{closed points of } A_k^1\} \xleftrightarrow{1:1} k.$$

5): Let k be an algebraically closed field, $A_k^2 = \text{Spec } k[x, y]$. It has a point η , corresponding to zero ideal, whose closure is the whole space.

$$\{\text{closed points of } A_k^2\} \xleftrightarrow{1:1} k^2$$

Also, for each irreducible polynomial $f(x, y) \in k[x, y]$, there is a point ξ corresponding to the $(f) \in k^2$, whose closure consists of ξ and $(a, b) \in k^2$ s.t. $f(a, b) = 0$.

Proposition: Let X be a scheme and A a ring, we have a canonical bijection,

$$\text{Hom}(X, \text{Spec } A) \cong \text{Hom}(A, \mathcal{O}_X(X)).$$

Proof: The assertion holds if $X = \text{Spec } B$ is affine, cover X by affine open subschemes,

$U_i = \text{Spec } B_i$, $i \in I$, given a homomorphism $\phi: A \rightarrow \mathcal{O}_X(X)$, composed by the restriction map $\mathcal{O}_X(X) \rightarrow \mathcal{O}_X(U_i) = B_i$, we get a homo $\phi_i: A \rightarrow B_i$, then ϕ_i is a morphism of affine schemes $f_i: \text{Spec } B_i \rightarrow \text{Spec } A$, to glue those f_i ($i \in I$)

to a morphism $f: X \rightarrow \text{Spec } A$, we now show $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, $U_i \cap U_j \rightarrow \text{Spec } A$. Cover $U_i \cap U_j$ by affine open schemes, $U_{ijk} = \text{Spec } B_{ijk}$, $k \in I_{ij}$, It remains to show

$$f_i|_{U_{ijk}} = f_j|_{U_{ijk}}, \quad \forall k \in I_{ij}$$

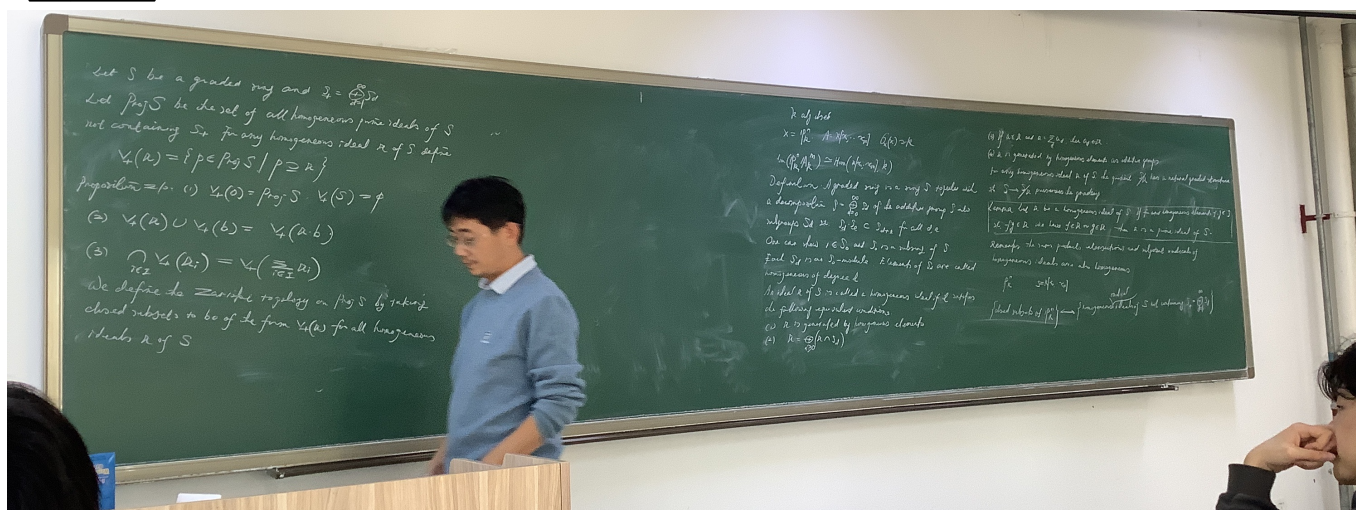
$$f_i|_{U_{ijk}} = (f_i|_{U_i \cap U_j})|_{U_{ijk}} = f_j|_{U_{ijk}}$$

We have $f_i|_{U_{ijk}}$ is the composition $U_{ijk} \xrightarrow{\phi''} \mathcal{O}_X(U_j) \xrightarrow{f_j} \text{Spec } A$. which is induced by homomorphism of rings:

$$A \xrightarrow{\phi} \mathcal{O}_X(X) \rightarrow \mathcal{O}_X(U_j) \rightarrow \mathcal{O}_X(U_{ijk})$$

$f_i|_{U_{ijk}}$ is induced by: $A \xrightarrow{\phi} \mathcal{O}_X(X) \rightarrow \mathcal{O}_X(U_j) \rightarrow \mathcal{O}_X(U_{ijk})$

Definition:



I've missed the part of the projective Schemes.

§3: Properties of schemes.

Proposition: Let A be a ring, then the following are equivalent.

1): $\text{Spec } A$ is connected.

2): A has no nilpotent elements other than 0.

3): For any decomposition $A = A_1 \times A_2$, we have $A_1 = 0$ or $A_2 = 0$.

Proof: 1) $\Rightarrow$ 2): Suppose $\text{Spec } A$ is connected, take $a \in A$, $a^2 = a$, $a(1-a) = 0$,

$$\text{Spec } A = V(aA) \cup V((1-a)A) \Rightarrow \text{Either } V(aA) \text{ or } V((1-a)A) \text{ is } \emptyset.$$

If $V(aA) = \text{Spec } A$, then a is nilpotent, $\Rightarrow a = 0$.

2) $\Rightarrow$ 3): $A = A_1 \times A_2$, $a \mapsto (1, 0)$, $(1-a) \mapsto (0, 1) \Rightarrow a = 0$ or 1 ,

$$\text{Say } a = 1, (1, 1) = (1, 0) \Rightarrow 1 = 0 \in A_2$$

3) $\Rightarrow$ 1): Suppose $\text{Spec } A$ is not connected, choose an open subset U of $\text{Spec } A$,

$U \neq \emptyset$, $U \neq \text{Spec } A$, $V = \text{Spec } A - U$ is also open. So $\text{Spec } A = U \cup V$.

$$A \cong \mathcal{O}_{\text{Spec } A}(\text{Spec } A) \cong \mathcal{O}_{\text{Spec } A}(U) \times \mathcal{O}_{\text{Spec } A}(V)$$

$$s \mapsto (s|_U, s|_V)$$

So, one of them is 0, i.e. $\mathcal{O}_{\text{Spec } A}(U) = 0$ or $\mathcal{O}_{\text{Spec } A}(V) = 0$,

$$\text{Saying } \mathcal{O}_{\text{Spec } A}(U) = 0, S: U \longrightarrow \coprod_{p \in \text{Spec } A} A_p, S(p) = 1 \in A_p, S \in \mathcal{O}_{\text{Spec } A}(U)$$

$$S': U \longrightarrow \coprod_{p \in \text{Spec } A} A_p, S'(p) = 0 \in A_p, S' \in \mathcal{O}_{\text{Spec } A}(U).$$

A desired contradiction.

Proposition: A closed subset of $\text{Spec } A$ is of the form $V(p)$ for a prime ideal p of A .

Proof: Let p be a prime ideal of A

Suppose $V(p) \subset V(a) \cup V(b)$ for some ideals a, b of A .

$$V(p) \subset V(ab) \Leftrightarrow p \supset ab \Rightarrow p \supset a \text{ or } p \supset b.$$

$$\Leftrightarrow V(p) \subset V(a) \text{ or } V(p) \subset V(b).$$

Proposition: Let X be a scheme, for any irreducible closed subset Y of X , there exists a unique point $\xi \in Y$, st: $Y = \overline{\{\xi\}}$, we call ξ the generic point of Y .

Proof: Take an open affine-schemes $U \cong \text{Spec } A$ of X , st: $U \cap Y \neq \emptyset$. Since Y is irreducible, then $Y \cap U$ is a closed irreducible subset of $U = \text{Spec } A$.

By the last proposition: $Y \cap U = V(p)$ for some $p \in \text{Spec } A$. Let $\xi \in U$ be the pt corresponding to $p \in \text{Spec } A$. then $\overline{\{\xi\}} \subset Y$, $\overline{\{\xi\}} \cap U = V(p) = Y \cap U$ dense in Y .

Let $\xi' \in Y$ st: $Y = \overline{\{\xi'\}}$, $\xi \in \overline{\{\xi'\}}$, $\xi \in U \Rightarrow \xi' \in U$, then:

$$V(p) = \overline{\{\xi\}} \cap U = \overline{\{\xi'\}} \cap U = V(p') \Rightarrow p = p', \text{ and hence } \xi = \xi'.$$

Corollary: Let A be a ring, st: $\text{Spec } A$ is a noetherian topological space, we have a bijection:

$$\left\{ \begin{array}{l} \text{irreducible closed} \\ \text{components of } \text{Spec } A \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{maximal prime} \\ \text{ideals of } A \end{array} \right\}.$$

Definition: A scheme $(X, \mathcal{O}_X)$ is called connected (irreducible) if X is connected, It is called an integral scheme if for any open subset U of X , $\mathcal{O}_X(U)$ is integral.

Proposition: Integral $\iff$ Irreducible + reduced.

Proof: Suppose $(X, \mathcal{O}_X)$ is integral, then it is reduced.

If X is not irreducible, then there are two non-empty subsets U and V of X , st: $U \cap V = \emptyset$, So, $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \times \mathcal{O}_X(V)$, is not integral.

Conversely, suppose X is irre and reduced. For any open set U of X , and $f_1, f_2 \in \mathcal{O}_X(U)$ st: $f_1 f_2 = 0$, $U_{f_i} = \{p \in U \mid \text{the germ of } f_i \text{ at } p \text{ is a unit in } \mathcal{O}_{X,p}\}$.

$U_{f_1} \cap U_{f_2} = \emptyset$. For any open affine subset, $V = \text{Spec } A$ of U , then

$f_i|_V \in \mathcal{O}_X(V)$ corresponds an element $g_i \in A$,

$$U_{f_i} \cap V = \left\{ p \in \text{Spec } A \mid g_i \text{ is a unit in } A_p \right\}.$$

$$= \left\{ p \in \text{Spec } A \mid g_i \notin p \right\} = D(g_i)$$

So U_{f_i} is an open subset of U . Since X is irreducible, $U_{f_1} = \emptyset$ or $U_{f_2} = \emptyset$,

Saying $U_{f_i} = \emptyset$, $U_{f_i} \cap V = D(g_i) = \emptyset \Leftrightarrow g_i$ is nilpotent in A . $\Rightarrow g_i = 0, f_i|_V = 0$,
 $\Rightarrow f = 0$

Proposition : A scheme X is reduced iff $\mathcal{O}_{X,p}$ is reduced for any $p \in X$.

Definition : Let $(X, \mathcal{O}_X)$ be an integral scheme, then there exists a unique point $\xi \in X$ such that $X = \overline{\{\xi\}}$. Call ξ the generic point of X . Then $\mathcal{O}_{X,\xi}$ is a field, which is called the function field of X .

$$p \in V \subset U, \quad \mathcal{O}_X(W) \hookrightarrow \mathcal{O}_X(U) \hookrightarrow \mathcal{O}_{X,p} \hookrightarrow \mathcal{O}_{X,\xi} \text{ a field.}$$

$$\parallel$$

$$\varinjlim_{p \in W} \mathcal{O}_{X(W)}.$$