Equivariant Cohomology

I. Borel Model

I.1. Some Attemptations

• Motivation The usual cohomology $H^k(X)$ of a manifold X encodes the topological data of X, if X endows with a Lie group G-action, then X also has the data of "G-Symmetricities", we want to find cohomology $H_G^k(X)$ which can also encodes the "Symmetrical" data of X. that kind of $H_G^k(X)$ will be called the equivariant cohomology of X.

• Attemptations: 1): How about $H_G^k(X) := H^k(X/G)$?

Nope! First, the quotient X/G is usually bad. Second, even for a compact Lie group action, $H^k(X/G)$ may lose a lot of information. For example, $S^1 \times S^2 \to S^2$ by rotation, the $S^2/S^1 = [0,1]$, it is a contractible, most of the $H^k([0,1])$ will vanish.

2): How about taking the cohomology of "G-invariant forms"?

Def: A form $\omega \in \Omega^k(X)$ is called G-invariant, if $\forall g \in G$, $g^*\omega = \omega$.

Since the pull-back g^* is a chain map for all $g \in G$, hence

$$g^* d\omega = d g^* \omega$$

hence the exterior differential d maps G-invariant forms to a G-invariant forms. We denote all G-invariant forms by $\Omega_G^k(X)$.

Def: The cohomology of "G-invariant forms" is the cohomology of the following Complex:

$$\cdots \longrightarrow \Omega_G^{k-1}(X) \xrightarrow{d} \Omega_G^k(X) \xrightarrow{d} \Omega_G^{k+1}(X) \longrightarrow \cdots$$

denoted by $H_G^k(X)$.

Back to the question. Nope! Since the $H_G^k(X)$ coincides with the de Rham Cohomology. if G is a compact connected Lie group

Indeed, there is an inclusion: $j: \Omega_G^k(X) \hookrightarrow \Omega^k(X)$, in particular it is a chain map.

Lemma: The inclusion j is a quasi-isomorphism, i.e it induces isomorphism:

If G is compact connected Lie group

$$j^*: H_G^k(X) \longrightarrow H^k(X;\mathbb{R})$$

Proof (Ex.6.4) It suffices to show j^* is surjective, i.e: $\forall [\beta] \in H^k(X;\mathbb{R})$, it contains at least 1 "G-invariant" closed form $\alpha \in [\beta]$.

Since G is compact connected, there exists a left invariant Haar measure on X, hence we can define:

$$\alpha = \int_G g^*\beta$$

a). α is closed:

$$d\alpha = d\int_G g^*\beta = \int_G dg^*\beta = \int_G g^* d\beta = 0.$$

b). $\alpha - \beta$ is exact, hence $\alpha \in [\beta]$.

Since g^* is an isomorphism, $g^*\beta \in [\beta]$, i.e

$$g^*\beta - \beta = d\omega_g$$

for some $\omega_g \in \Omega^{k-1}(X)$, hence

$$\alpha - \beta = \int_G g^*\beta - \beta = d\int_G \omega_g$$

which is exact.

c). α is G-invariant. In fact, by changing of the variable:

$\forall h \in G$,

$$h^*\alpha = \int_G h^*g^*\beta = \int_G (gh)^*\beta = \alpha.$$

Hence α is a desired form, and j^* is an isomorphism.

• Remark: The Attemptation 2) is about to success, if we take "G-equivariant" instead of "G-invariant", which is the Cartan's model for equivariant cohomology, the equivariant de Rham Cohomology. Which coincides with the Borel model which we'll discuss lately.

I.2. Borel Construction

We know from previous classes, G acts freely on EG, hence the G-action:

$$\begin{aligned} G\times(EG\times X) &\longrightarrow EG\times X \\ g\cdot(e,x) &\longmapsto (ge, g\cdot x) \end{aligned}$$

is also free, hence we can define:

Def 1.1: $X_G := EG\times X/G$ is called the Borel construction of X.

• Remark: 1). X_G is a fiber bundle over $EG/G = BG$ with fibre X. $(\pi: X_G \to BG)$

2). However, usually the X_G is not a fibration over X/G.

We notice that, we have the following commutative diagram:

$$\begin{tikzcd}
(e,x) \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & x \arrow[ddd, mapsto] \\
& EG \times X \arrow[r, "Prj_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "\pi_2"] & \\
& X_G = EG \times X / G \arrow[r, dashed, "\sigma"] & X/G & \\
{[e,x]} \arrow[rrr, mapsto] & & & {[x]}
\end{tikzcd}$$

It is not hard to check σ is well-defined.

Aspite σ may not be a fibration, but:

Theorem I.1: If G acts freely on X, then $\sigma : X_G \longrightarrow X/G$ is a fibration with a contractible fibre EG, hence X_G is homotopy equivalent to X/G.

Proof: $\forall [x] \in X/G$, we have

$$\begin{aligned}
\sigma^{-1}[x] &= \{ [e,y] \in X_G \mid [y] = [x] \} \\
&= \{ [e, g.x] \in X_G \mid g \in G \} \\
&= \{ [g^{-1}e, x] \in X_G \mid g^{-1} \in G \} \\
&= EG \times \{x\} / G_x = EG / G_x
\end{aligned}$$

Since G acts freely on X, hence $\sigma^{-1}[x] = EG$.

Def 1.2: We define the Borel model of equivariant Cohomology of X as:

$$H_G^k(X) := H^k(X_G)$$

There are some remarks of the equivariant cohomology ring.

Remark: 1): If X is a point, then

$$X_G = EG \times \{pt\} / G = EG/G = BG.$$

hence $H^k_G(pt) = H^k(BG)$, the cohomology ring of which is never zero.

2). Since we have $\pi: X_G \longrightarrow BG$, it induces ring morphism:

$$\begin{array}{ccc} \pi^*: H^*(BG) & \longrightarrow & H^*(X_G) \\ \| & & \| \\ H^*_G(pt) & & H^*_G(X) \end{array}$$

usually this is a torsion $H^*_G(pt)$-module, it will be interesting to investigate when the torsion is free?

where $H^* = \bigoplus_{k \geq 0} H^k$ represents the cohomology ring.

hence $H^*_G(X)$ has a $H^*_G(pt)$-module structure, so we need to know what is $H^*(BG)$, especially for G is a torus.

3): For $G = S^1$, we know $BS^1 = \mathbb{P}^\infty$, and: (from Hatcher)

$$H^*(\mathbb{P}^\infty; \mathbb{Z}) \cong \mathbb{Z}[u].$$

where u is a generator of deg 2, i.e: $u \in H^2(\mathbb{P}^1; \mathbb{Z})$

From Künneth Formula: $T^n = S^1 \times \cdots \times S^1$

$$H^*(BT^n; \mathbb{Z}) = H^*(BS^1) \otimes \cdots \otimes H^*(BS^1)$$
$$= \mathbb{Z}[u_1, \cdots, u_n].$$

4): If we take $\mathbb{R}$-coefficient, the generator u of $H^*(\mathbb{P}^\infty; \mathbb{R})$ is the Fubini-Study form ω_{FS} on $\mathbb{P}^1$, since it is closed, not exact, and

$$\langle \omega_{FS}, \mathbb{P}^1 \rangle = \int_{\mathbb{P}^1} \omega_{FS} = 1$$

II. Euler Class

then $p: X \to X/S^1$ is a principal S^1- bundle.

We consider the free S^1- action $S^1 \times X \to X$, the fibration $\sigma: X_{S^1} \to X/S^1$ (where $X_{S^1} = X \times_{S^1} / S^1$) induces isomorphism of cohomology ring:

$$\sigma^*: H^*(X/S^1) \longrightarrow H^*(X_{S^1}) = H^*_{S^1}(X)$$

Recall that $H^*_{S^1}(X)$ is a $H^*(BS^1; \mathbb{Z}) = \mathbb{Z}[u]$ - module.

Def 2.1: The class $e \in H^2(X/S^1)$ mapping to $-u$ by σ^* is called the Euler class of S^1- bundle $(p, X, X/S^1)$.

Remark: Euler class e is "natural," in the sense that:

if $f: M \longrightarrow X/S^1$ a smooth map between base spaces of the S^1- bundles X, and the pull-back bundle $f^\#(X)$, then:

$$f^* e = e_M$$

where e_M is the Euler class of M.

Proof: We have the following diagram:

$$\begin{tikzcd}
f^\#(X) \arrow[r] \arrow[d, "p^\#"'] & X \arrow[d, "p"] \\
M \arrow[r, "f"'] & X/S^1
\end{tikzcd}$$

hence we have diagram:

$$\begin{tikzcd}
H^*_{S^1}(f^\#(X)) & H^*_{S^1}(X) \arrow[l] \\
H^*(M) \arrow[u] & H^*(X/S^1) \arrow[l, "f^*"'] \arrow[u]
\end{tikzcd}$$

Example: The principal S^1-bundle

$$S^{2n+1} \longrightarrow \mathbb{P}^n = S^{2n+1}/S^1$$

The Euler class in $H^2(\mathbb{P}^n; \mathbb{Z})$ is exactly $-u$.

In fact, it will be easy to consider in $\mathbb{R}$-coefficient, the u lies in $H^2(\mathbb{P}^1; \mathbb{Z}) \subset H^*_{S^1}(S^{2n+1})$ is the Fubini-Study form on $\mathbb{P}^1$.

Then

$$\int_{S^{2n+1}/S^1} \sigma^*(-u) = \int_{\sigma(S^{2n+1}/S^1)} -u = \int_{\mathbb{P}^n} -u = -1$$

Theorem 2.1 Let $p: X \to X/S^1$ be a principal S^1-bundle, let η be a curvature form on X, then the cohomology class of the form $\frac{\eta}{2\pi} \in H^2(X/S^1; \mathbb{R})$ is the Euler class of the bundle.

Proof: Recall that, from previous chapter, we know the curvature form class does not depend on a S^1-bundle on the choice of the connections.

Since $[B; \underset{\mathbb{P}^\infty}{BS^1}] \xleftrightarrow{1-1} \mathrm{Prin}_{S^1}(B)$, hence for our $(p, X, X/S^1)$, $\exists\, n$ big enough, s.t. $(p, X, X/S^1)$ is the pull-back bundle of

$$S^{2n+1} \longrightarrow \mathbb{P}^n.$$

under the classifying map: $f: X/S^1 \to \mathbb{P}^n$.

hence we have the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{F} & S^{2n+1} \\ \downarrow & & \downarrow \\ X/S^1 & \xrightarrow{f} & \mathbb{P}^n \end{array}$$

by the naturality of Euler class:

$$e_{X/S^1} = f^*(-u)$$

which is again a class of curvature form.

Remark: From now on, we can see that, the Euler class of an S^1-bundle is the generalization of the 1st Chern class $c_1(L)$ of a complex line bundle.

The 1st Chern class is defined as the class of the curvature form associated to an Hermitian metric on the line bundle L.