Reading Notes on Kodaira Embedding Theorem

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Abstract

This is my final homework of the course "Linear Systems in Algebraic Varieties", the homework is mainly the reading notes of the Kodaira embedding theorem, which states as:

Theorem 0.1 (Kodaira)**.** *Let X be a compact Kähler manifold, if X endowed with a positive line bundle, then it can be embedded to some projective space:*

$$i : X \hookrightarrow \mathbb{P}^n$$

As another part of this homework, I investigated a type of ruled surface, the Hirzbruch surfaces, and I gave a diffeomorphic classification of them, it will be appeared in the appendix.

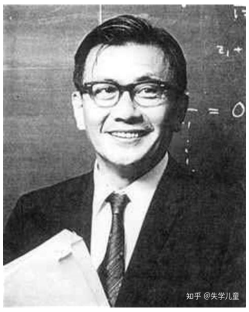

Figure 1: Kunihiko Kodaira

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1 Ampleness, Very Ampleness of a Line Bundle

1.1 Basic Definitions

Let X be a compact manifold, and $\pi : L \longrightarrow X$ is a holomorphic line bundle.

Definition 1.1 (Spanned Line Bundle)**.** *We say a line bundle L is **spanned**, if for all $x \in X$, there exists a section $s \in H^0(X, L)$ such that $s(x) \neq 0$.*

Example. For $\mathbb{P}^n$, the line bundle $\mathcal{O}(n)$ is spanned if and only if $n \geq 0$.

Remarks.
(1). If a point $x \in X$ so that all sections vanishing at this point, such a point will be called a **base point** of the line bundle L, the collection of all base points of L is called the **base point locus** of L, denoted by $BS(L)$.
(2). If we choose a local trivialization $\{U_\alpha, \phi_\alpha\}_{\alpha\in\Lambda}$ of L, where $\phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}^1$, a section $s \in H^0(X, L)$ can be expressed locally by

$$s_\alpha := \phi_\alpha \circ s : U_\alpha \longrightarrow \mathbb{C}$$

It is clearly that $s_\alpha \in \mathcal{O}(U_\alpha)$, and if for some $U_\beta \cap U_\alpha \neq \varnothing$, one has

$$s_\alpha(x) = g_{\alpha\beta}(x)s_\beta(x)$$

on $U_\alpha \cap U_\beta$, where $g_{\alpha\beta}$ is the clutching function and it is in $\mathcal{O}^*(U_\alpha \cap U_\beta)$.

Definition 1.2. *For a spanned line bundle L, we define a map:*

$$i_L : X \longrightarrow \mathbb{P}(H^0(X, L))$$

$$x \mapsto H_x$$

where H_x is hyperplane of $H^0(X, L)$ consisting of all global sections which vanishing at $x \in X$.

Remarks.
(1).We need recall that the space of global sections of a line bundle $\Gamma(L)$ is a finite generated $\mathcal{O}(X)-$module, since our X is a compact manifold, the global holomorphic functions are constants, hence it is a $\mathbb{C}-$space, hence the projectivization of $H^0(X, L)$ is exactly $\mathbb{P}^n$, for some integer n.
(2).The map ϕ_L can be expressed locally, if we choose a local trivialization $\{U_\alpha, \psi_\alpha\}_{\alpha\in\Lambda}$, as the notation in the last remark, denoted by

$$s^i_\alpha := \psi_\alpha \circ s^i : U_\alpha \longrightarrow \mathbb{C}$$

the local expression of the $i-$th basis of global sections, then we have

$$\phi_L|_{U_\alpha}(x) = [s^0_\alpha(x), ..., s^n_\alpha(x)]$$

and the expression form will not be change when changing the local trivialization, since we have

$$\begin{aligned}\phi_L|_{U_\alpha\cap U_\beta}(x) &= [s^0_\alpha(x), ..., s^n_\alpha(x)]\\ &= [g_{\alpha\beta}(x)s^0_\beta(x), ..., g_{\alpha\beta}(x)s^n_\beta(x)]\\ &= [s^0_\beta(x), ..., s^n_\beta(x)]\end{aligned}$$

and it is well-defined since the line bundle is spanned.

[1]Sometimes I will use $L|_{U_\alpha}$ instead of the notation $\pi^{-1}(U_\alpha)$

(3).It is not hard to see that the ϕ_L is holomorphic.
(4)Moreover, we have the pull-back:

$$\phi_L^*(\mathcal{O}_{\mathbb{P}^n}(1)) = L$$

Indeed, let the z_0 be a global section of $\mathcal{O}_{\mathbb{P}^n}(1)$, the divisor associated to it is denoted by D_0, then the pull-back of D_0 under $\phi|_L$ is s^0, which corresponds to the line bundle L over X, hence

$$\phi_L^*(\mathcal{O}_{\mathbb{P}^n}(1)) = \phi_L^*(D_0) = L$$

Definition 1.3 (Ampleness, Very Ampleness)**.** *A line bundle L is **very ample**, if $\phi_L : X \longrightarrow \mathbb{P}^n$ is an embedding, it is **ample**, if there exists some positive integer $k > 0$ such that $L^{\otimes k}$ is very ample, a divisor D is said to be (very) ample if the corresponding line bundle $\mathcal{O}(D)$ is (very) ample.*

Example. The line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ is very ample.

1.2 Cohomological Characterization of Very Ampleness

In this subsection, we will describe when ϕ_L is an embedding in terms of the language of cohomology, recall that an embedding is an injective immersion.
(1). First, $\phi_L : X \longrightarrow \mathbb{P}^n$ need to be well-defined, i.e. the line bundle need to be spanned. If we denoted by

$$A = \{s(x) | s \in H^0(X, L), x \in X\}$$

and define the skyscraper sheaf $\mathcal{L}_x$ as

$$\mathcal{L}_x(U) = \begin{cases} A & x \in U \\ 0 & x \notin U \end{cases}$$

hence , this condition it is equivalent to say the restriction map:

$$H^0(X, L) \xrightarrow{|_x} H^0(X, \mathcal{L}_x)$$

is surjective. This map is sited in the long exact sequence induced by the short exact sequence:

$$0 \longrightarrow L \otimes \mathcal{I}_x \longrightarrow L \longrightarrow \mathcal{L}_x \longrightarrow 0$$

Hence the condition "well-define" is equivalent to

$$H^1(X, L \otimes \mathcal{I}_x) = 0$$

(2). Secondly, the ϕ_L need to be injective, that is for $x \neq y \in X$, there exists a section $s \in H^0(X, L)$ which vanishes at x but not at y, like (1), it is equivalent to

$$H^0(X, L) \xrightarrow{|_{x,y}} H^0(X, \mathcal{L}_x \oplus \mathcal{L}_y)$$

is surjective, it is sited in the long exact sequence induced by the short exact sequence:

$$0 \longrightarrow L \otimes \mathcal{I}_{x,y} \longrightarrow L \longrightarrow \mathcal{L}_x \oplus \mathcal{L}_y \longrightarrow 0$$

Hence the injectiveness is equivalent to

$$H^1(X, \mathcal{I}_{x,y}) = 0$$

(3). Finally, the map ϕ_L need to be an immersion, so we need to check the differential map

$$d(\phi_L)_x : T_x^{(1,0)} X \longrightarrow T_{\phi_L(x)} \mathbb{P}^n$$

If we choose basis $s_0, s_1, ..., s_n$ of $H^0(X, L)$ in some local trivialization, and assume that $s_0(x) \neq 0$, then the map ϕ_L can be locally write as

$$\phi_L(x) = \left[\left(\frac{s_1(x)}{s_0(x)} \right), ..., \left(\frac{s_n(x)}{s_0(x)} \right) \right]$$

hence $(d\phi_L)_x$ is injective if and only if $d\left(\frac{s_1}{s_0}\right)_x, ..., d\left(\frac{s_n}{s_0}\right)_x$ spanned the cotangent space $T_x^{*(1,0)}X$, which is equivalent to say

$$d_x : H^0(X, L) \longrightarrow H^0(X, \mathcal{L}_x \otimes T_x^{*(1,0)}X)$$
$$s_x \mapsto (ds)_x$$

is surjective, since

$$\mathcal{I}_x / \mathcal{I}_x^2 = T_x^{*(1,0)}X$$

so it is sited in the long exact sequence induced by the short exact sequence

$$0 \longrightarrow L \otimes \mathcal{I}_x^2 \longrightarrow L \longrightarrow \mathcal{L}_x \otimes T_x^{*(1,0)}X \longrightarrow 0$$

So, in all, we have

Theorem 1.1. *A line bundle L is very ample if and only if*

$$H^1(X, L \otimes \mathcal{I}_x^2) = H^1(X, \mathcal{I}_{x,y}) = 0$$

for all $x, y \in X$

2 Positivity in Complex Geometry

2.1 Basic Complex Analytic Geometry

Recall. (1).From now on we assume (X, ω) is a compact Kähler manifold, where ω is the image part of the Hermitian metric h, which is called the **Kähler form**, it is a non-degenerate closed 2-form[2], and we use (L, h) to represent for a line bundle endowed with an Hermitian metric h.
(2).That h has a local expression under a local trivialization $\{U_\alpha, \psi_\alpha\}$ of the line bundle L, and

$$h_\alpha := h|_{U_\alpha} = \psi_\alpha \circ h \in \mathcal{O}(U_\alpha)$$

and for some $U_\alpha \cap U_\beta \neq \varnothing$, one has

$$h_\alpha(x) = |g_{\alpha\beta}(x)|^2 h_\beta(x)$$

for some $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$.
(3). For any Hermitian metric h, there exists a Chern-Levi-Civita connection

$$\nabla : \Gamma(L) \longrightarrow \Gamma(T^*X \otimes L)$$

on the line bundle L, usually, the connection ∇ is a matrix-valued 1-form:

$$\nabla = d + A$$

[2]People prefer to call such a 2-form a **symplectic form**.

since our bundle is a line bundle, hence the connection matrix A is a precisely complex 1-form, the curvature form is defined by

$$\Omega_\nabla = dA + A \wedge A \in \Omega^{(1,1)}(X;\mathbb{C})$$

which is a complex $(1,1)-$form.
(4). As above, the connection ∇ and the curvature Ω_∇ both have the local expression, we use notation in (1), and we shall use the Dolbeault operator $d = \partial + \overline{\partial}$, then

$$\nabla|_{U_\alpha} = \partial + \bar{\partial} + \partial \log h_\alpha$$

$$\Omega_\nabla|_{U_\alpha} = \bar{\partial}\partial \log h_\alpha$$

In particular, we can see that the curvature form Ω_∇ is a closed form, hence it is a cocycle in the Dolbeault-de Rham cohomology group.

Definition 2.1 (1st Chern Class)**.** *The class*

$$c_1(L) = \left[\frac{i}{2\pi}\Omega_\nabla\right] \in H^2(X)$$

is called the ***1st Chern class*** *of the line bundle L.*

Theorem 2.1. *The 1st Chern class does not depend on the choice of the connections!*

proof. In the notations of the previous recall, given any two Hermitian metrics h_1, h_2 on L, with curvature form Ω_1, Ω_2 respectively, the quotient

$$\frac{h_1}{h_2} := \frac{h_1^\alpha}{h_2^\beta}$$

is independent of the trivialization $\{U_\alpha, \psi_\alpha\}_{\alpha\in\Lambda}$ of L, thus it is a well-defined positive function e^ρ for some real smooth function ρ, the formula $h_2 = e^\rho h_1$ yields

$$\Omega_2 = \bar{\partial}\partial\rho + \Omega_1$$

hence we have

$$\left[\frac{i}{2\pi}\Omega_1\right] = \left[\frac{i}{2\pi}\Omega_2\right] \qquad \blacksquare$$

Remark. There is another more algebraical way to define the Chern class.
We denoted by $\mathcal{O}_X^*$ the sheaf of holomorphic functions without zeros, and $\mathbb{Z}$ the constant sheaf, then we have the short exact sequence of sheaves:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

it will induce the long exact sequence on the level of sheaf cohomology groups, in particular, we will have the connecting boundary:

$$\delta^* : H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z})$$

the later cohomology group is naturally isomorphic to the 2nd de Rham cohomology with coefficient in $\mathbb{Z}$, and since

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X)$$

we can define the Chern class $c_1(L)$ of a line bundle $L \in \mathrm{Pic}(X)$ as the image under δ^*, i.e.

$$c_1(L) := \delta^*(L) \in H^2(X, \mathbb{Z})$$

Example. For $X = \mathbb{P}^1$, we have $H^2(\mathbb{P}^1, \mathbb{Z}) \cong \mathbb{Z}$, the Chern class of the line bundle $\mathcal{O}(n)$ is just n.

2.2 Positivity of a Line Bundle

Definition 2.2. *We say a real* $(1,1)-$*form* ω *on* X *is positive if for any* $x \in X$ *and all non-zero* $v \in (T_x X)_{\mathbb{R}}$*, one has*

$$\omega(v, J_x v) > 0$$

where J_x *is the almost complex structure induced by the complex structure on* X*.*

Definition 2.3 (Positivity)**.** *A line bundle L is* ***positive*** *if there exists a metric h on L such that the curvature form* Ω_h *is a positive (1,1)-form.*

Theorem 2.2. *A line bundle L is positive if and only if its 1st Chern class can be represented by a positive form in* $H^2(X)$*.*

For the detailed proof, see [1].

Example. The line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ is positive.
In fact, we shall first do this on its dual bundle, as the notation used above, we assume a local trivialization on U_α of $\mathcal{O}_{\mathbb{P}^n}(-1)$ is given by

$$\psi_\alpha := ([z_0, ..., z_n], z_\alpha)$$

now, define the Hermitian metric on $\mathcal{O}_{\mathbb{P}^n}(-1)$ locally via

$$h_\alpha = \frac{1}{z_\alpha} \sum_{i=0}^{n} |z_i|^2$$

Then the curvature form on $\mathcal{O}_{\mathbb{P}^n}(-1)$, denoted by Ω^* is given by

$$\begin{aligned} \Omega^*|_{U_\alpha} &= \bar{\partial}\partial \log \left(\frac{1}{|z_\alpha|^2} \sum_{i=0}^{n} |z_i|^2 \right) \\ &= \bar{\partial}\partial \log \left(\sum_{i=0}^{n} |z_i|^2 \right) \end{aligned}$$

Then the curvature form on $\mathcal{O}_{\mathbb{P}^n}(1)$, denoted by Ω, is $-\Omega^*$, hence

$$\begin{aligned} c_1(\mathcal{O}_{\mathbb{P}^n}(1)) &= \left[-\frac{i}{2\pi} \bar{\partial}\partial \log \left(\sum_{i=0}^{n} |z_i|^2 \right) \right] \\ &= \left[d\bar{d} \log \left(\sum_{i=0}^{n} |z_i|^2 \right) \right] \end{aligned}$$

which is the (1,1)-form associated to the Fubini-Study, metric on $\mathbb{P}^n$ and hence positive.

Theorem 2.3. *On a compact Kähler manifold X, any ample line bundle L is positive.*

proof. Since L is ample, there exists some integer $k > 0$ such that $L^{\otimes k}$ is very ample, i.e. the mapping

$$\phi_{L^{\otimes k}} : X \hookrightarrow \mathbb{P}^n$$

is embedding, since $\mathcal{O}_{\mathbb{P}^n}(1)$ is positive, hence there exists a positive Hermitian metric on it, and the pull-back metric gives rise to a positive Hermitian metric on $L^{\otimes k}$, and the $k-$th root metric will give a desired positive metric. ∎

Conversely, any positive line bundle will be ample, this is the part of the Kodaira embedding theorem.

2.3 Kodaira Vanishing Theorem

Theorem 2.4 (Kodaira-Akizuki-Nakani Vanishing Theorem)**.** *If L is a positive line bundle, on a complex compact manifold X, then for all $p+q>0$, we have*

$$H^{(p,q)}(X,L) = H^q(X,\Omega_X^p \otimes L) = 0$$

In particular:

$$H^q(X,K_X \otimes L) = 0$$

for all $q>0$, where K_X is the canonical line bundle over X.

For a detailed proof, I'd like to refer [1].
As an application, there is a low-dimensional version of the Kodaira embedding theorem:

Theorem 2.5. *Every compact Riemann surface can be embedded to a projective space.*

proof. We will mainly show that for any divisor D with degree $\deg D \geq 2g+1$ on a compact Riemann surface X with genus g is very ample. Theorem 2 will be a powerful tool.
We notice that

$$H^1(X,\mathcal{O}_X(D)\otimes \mathcal{I}_x^2) = H^1(X, D-2[x]) = H^1(X, K_X + (K_X - 2[x] - K_X))$$

and since

$$\deg(D-2[x]-K_X) = \deg D - 2 - 2g + 2 \geq 1$$

the line bundle $\mathcal{O}_X(D-2[x]-K_X)$ is positive, thus by Kodaira vanishing theorem,we have:

$$H^1(X,\mathcal{O}_X(D)\otimes \mathcal{I}_x^2) = 0$$

Analogously we can show $H^1(X,\mathcal{O}_X(D)\otimes \mathcal{I}_x) = 0$, hence by theorem 2, D is very ample. ∎

However, for high-dimensional manifold X, the ideal sheaf $\mathcal{I}_x$ may not be an invertible sheaf, this method lost its power on this case, but we can replace the point by a divisor, the so called the exceptional divisor after a topological surgery: the blowing up.

3 Blowing Up

We will mainly discuss about the blowup $\tilde{X}$ at a point $x \in X$, which will be the main technique appeared in the proof of Kodaira embedding theorem.

3.1 Blow Up at a Point

We first start from $\mathbb{C}^n$, let U be a neighbourhood of 0 in $\mathbb{C}^n$ with local coordinate $z_1, ..., z_n$.

Definition 3.1. *Define:*

$$\tilde{U} = \{(z,\ell) \in U \times \mathbb{P}^{n-1} | z_i \ell_j = z_j \ell_i\}$$

with a projection:

$$\pi : \tilde{U} \longrightarrow U$$
$$(z,\ell) \mapsto z$$

Remark. $\tilde{U}$ has some equivalent characterizations:

$$\tilde{U} = \left\{ (z,\ell) \in U \times \mathbb{P}^{n-1} \middle| \operatorname{rank}\begin{pmatrix} z_1 & \dots & z_n \\ \ell_1 & \dots & \ell_n \end{pmatrix} \leq 1 \right\}$$
$$= \{(z,\ell) \in U \times \mathbb{P}^{n-1} | z = (z_1, ..., z_n) \in \ell = [\ell_1, ..., \ell_n]\}$$

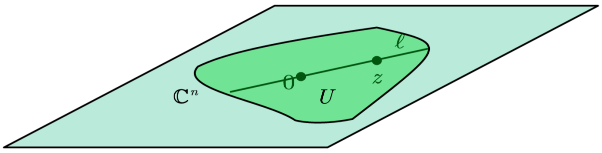


Figure 2: $\tilde{U}$

Proposition 1. *By our definition we have*
(1). $E := \pi^{-1}(0) \cong \mathbb{P}^{n-1}$, *which is called the exceptional divisor.*[3]
(2). *The restriction of the projection:*

$$\pi|_{\tilde{U}\setminus E} : \tilde{U} \setminus E \xrightarrow{\cong} U \setminus \{0\}$$

is a biholomorphism.

Definition 3.2. *If we denoted by* $\varphi := \pi|_{\tilde{U}\setminus E}$, *we call*

$$\widetilde{\mathbb{C}^n} := (\mathbb{C}^n \setminus \{0\} \coprod \tilde{U} \setminus E) \,/ \sim_\varphi$$

the ***blowing up*** *of* $\mathbb{C}^n$ *at the origin.*

Proposition 2. *This definition is independent of the Choice of* U, *hence it is well-defined.*

proof. For $(z'_1, ..., z'_n) := (f_1(z_1), ..., f_n(z_n))$, we can see that the diffeomorphism

$$f : \tilde{U} \setminus E \longrightarrow \tilde{U}' \setminus E'$$

may be extended via

$$f(0, \ell) = (0, \ell') \qquad \ell'_i = \sum_{k=1}^{n} \frac{\partial f_i}{\partial z_k}\Big|_0 \ell_k \qquad \blacksquare$$

<u>**Remark.**</u> (1). It is possible to define the blowup at a point x on any complex manifold X.
(2).We can see that the exceptional divisor E can be identified with $\mathbb{P}(T_x^{(1,0)}X)$, via

$$(0, \ell) \mapsto \left[\sum_{k=1}^{n} \ell_k \frac{\partial}{\partial z_k}\right]$$

(3).Next, we will describe the local coordinate of the blown up.
On $\tilde{U} = \{(z, \ell) \in U \times \mathbb{P}^{n-1} | z_i\ell_j = z_j\ell_i\}$,we define the local charts $\tilde{U}_i := \tilde{U} \setminus \{\ell_i = 0\}$, we define:

$$\begin{aligned} \varphi_i : \tilde{U}_i &\longrightarrow \mathbb{C}^n \\ (z, \ell) &\mapsto \left(\frac{z_1}{z_i}, ..., \frac{z_{i-1}}{z_i}, z_i, ..., \frac{z_n}{z_i}\right) \\ &:= (z_1^i, ..., z_n^i) \end{aligned}$$

we have the coordinate transformation:

[3]This divisor is defined as the Weil divisor, where E is a hypersurface in $\tilde{X}$, the manifold after the blowup, the divisor take value 1 at E and zero and any other hypersurfaces.

$$\varphi_i \circ \varphi_i^{-1}|_{U_i \cap U_j}(z_1^i, ..., z_n^i) = \left(\frac{z_1^i}{z_j^i}, ..., \frac{z_{j-1}^i}{z_j^i}, z_j^i z_i^i, ..., \frac{1}{z_j^i}, ..., \frac{z_n^i}{z_j^i} \right)$$

Hence locally we have

$$\pi|_{\tilde{U}_i}(z_1^i, ..., z_n^i) = (z_i z_1^i, ..., z_i z_n^i)$$
$$D_E|_{\tilde{U}_i} = (z_i)$$

Since the exceptional divisor $D_E|_{\tilde{U}_i} = (z_i)$, hence the transition function of the line bundle $\mathcal{O}_{\tilde{X}}(E)$ is given by

$$g_{ij} = \frac{z_i}{z_j} = \frac{\ell_i}{\ell_j} : \tilde{U}_i \cap \tilde{U}_j \longrightarrow \mathbb{C}$$

So we can realize the line bundle $\mathcal{O}_{\tilde{U}}(E)$ by identifying the fiber at point (z, ℓ) as the complex line passing through $(\ell_1, ..., \ell_n)$, more particularly, the restriction of this line bundle on E is exactly the tautological line bundle

$$\mathcal{O}_E(E) = \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$$

(4). Under our identification $\mathbb{P}(T_x^{(1,0)}X) = E$, we will have

$$H^0(E, \mathcal{O}_E(-E)) = T_x^{*(1,0)}X$$

Moreover, we have the following diagram commutes:

$$\begin{array}{ccc} H^0(U, \mathcal{I}_x) & \xrightarrow{\quad \pi^* \quad} & H^0(\tilde{U}, \mathcal{O}_{\tilde{U}}(-E)) \\ & {\scriptstyle d} \searrow \quad \swarrow {\scriptstyle |_E} & \\ & T_x^{*(1,0)}X = H^0(E, \mathcal{O}_E(-E)) & \end{array}$$

In the viewpoint stated in [7], the local analytic behavior of a function at x is magnified to the global behavior of $\tilde{X}$.

3.2 Line Bundles on a Blown Up

Theorem 3.1. *If L is a positive line bundle over a complex manifold X, then there exists a positive integer $k > 0$ such that for any $n \in \mathbb{Z}$ the line bundle $\pi^* L^k \otimes \mathcal{O}_{\tilde{X}}(-nE)$ is a positive line bundle.*

proof. We first construct a metric on $\mathcal{O}_{\tilde{X}}(E)$ by the uniform decomposition.
(1).Let h_1 be the metric on $\mathcal{O}_{\tilde{U}}(E)$ restriction of the standard metric in $\mathbb{C}^n$ passing through $(\ell_1, ..., \ell_n)$.
(2).Let h_2 be the metric on $\mathcal{O}_{\tilde{X} \setminus E}(E)$ such that $h_2(\sigma) \equiv 1$, where $\sigma \in H^0(\tilde{X}, E)$ is a global section of $\mathcal{O}_{\tilde{E}}(E)$ with $(\sigma) = E$.
(3). For $\epsilon > 0$, $U_\epsilon := \{z \in U \,||\, ||z|| < \epsilon\}$ and $\tilde{U}_\epsilon := \pi^{-1}(U_\epsilon)$. Let ρ_1, ρ_2 be a partition of unity to the cover $\{\tilde{U}_{2\epsilon}, \tilde{X} \setminus \tilde{U}_\epsilon\}$ of X and h be a global Hermitian metric given by

$$h = \rho_1 h_1 + \rho_2 h_2$$

Then let's compute the positivity of this metric.
(a). On $\tilde{X} \setminus \tilde{U}_{2\epsilon}$, $\rho_2 \equiv 1$, hence $h_2(\sigma) \equiv 1$, i.e. in the trivialization above $h_\alpha |\sigma_\alpha|^2 = 1$, and

$$c_1(E) = -d\bar{d} \log \frac{1}{|\sigma|^2} = 0$$

since $\log 1/|\sigma|^2$ is harmonic.
(b). On $\tilde{X} \setminus \tilde{U}_{2\epsilon}$, $\rho_2 \equiv 0$, we denote

$$\pi' : \tilde{U} \longrightarrow \mathbb{P}^{n-1}$$

$$(z, \ell) \mapsto \ell$$

then

$$c_1(E) = d\bar{d} \log ||z||^2 = -(\pi')^* \omega_{FS}$$

hence the pull-back $\pi'^* \omega_{FS}$ of the fundamental (1,1)-form associated to the Fubini-Study metric under the map π' and $c_1(E)$ is semi-positive on $\tilde{U}_\epsilon \setminus E$.
(c). On E, we have, by continuity form previous remark:

$$-c_1(E)|_E = \omega > 0$$

So, to sum up:

$$c_1(-E) = \begin{cases} 0 & \tilde{X} \setminus \tilde{U}_{2\epsilon} \\ \geq 0 & \tilde{U}_\epsilon \\ > 0 & T_x^{(1,0)} E \subset T_x^{(1,0)} \tilde{X} \end{cases}$$

Let (L, h_L) be an Hermitian positive line bundle over $\tilde{X}$, then

$$c_1(\pi^* L) = \pi^* c_1(L)$$

For any $x \in E$ and $v \in T_x \tilde{X}$ we have

$$c_1(\pi^* L)(v, \bar{v}) = c_1(L)(\pi^* v, \overline{\pi^* v}) \geq 0$$

where the equality holds if and only if $\pi^* v = 0$. Hence we have

$$c_1(\pi^* L) = \begin{cases} \geq 0 & \text{everywhere} \\ > 0 & \tilde{X} \setminus E \\ > 0 & T_x^{(1,0)} \tilde{X} / T_x^{(1,0)} E \end{cases}$$

So we can see that

$$c_1(\pi^* L^k \otimes (-E)) = k c_1(\pi^* L) - c_1(E)$$

is positive on $\tilde{U}_\epsilon$ and $\tilde{X} \setminus \tilde{U}_{2\epsilon}$ for ϵ small enough. Since $\tilde{U}_{2\epsilon} \setminus \tilde{U}_\epsilon$ is relatively compact, $-c_1(E)$ is bounded below and $c_1(\pi^* L)$ is strictly positive, then for k big enough, $\pi^* L^k \otimes \mathcal{O}_{\tilde{X}}(-nE)$ will be a positive line bundle. ■

Theorem 3.2. *Let K_X denote the canonical line bundle over X, we have*

$$K_{\tilde{X}} = \pi^*(K_X) \otimes \mathcal{O}_{\tilde{X}}((n-1)E)$$

For a detailed proof I'd like to refer to [7].

4 Kodaira Embedding Theorem

Theorem 4.1 (Kodaira Embedding). *A line bundle L over a compact Kähler manifold X is positive if and only of it is ample, hence if X admits a positive line bundle, then X is projective.*

proof. The ampleness implies the positivity is stated in theorem 2.3, so it suffices to show if L is positive, then there exists some integer $k > 0$ such that

$$\phi_{L^k} : X \hookrightarrow \mathbb{P}(H^0(X, L^k))$$

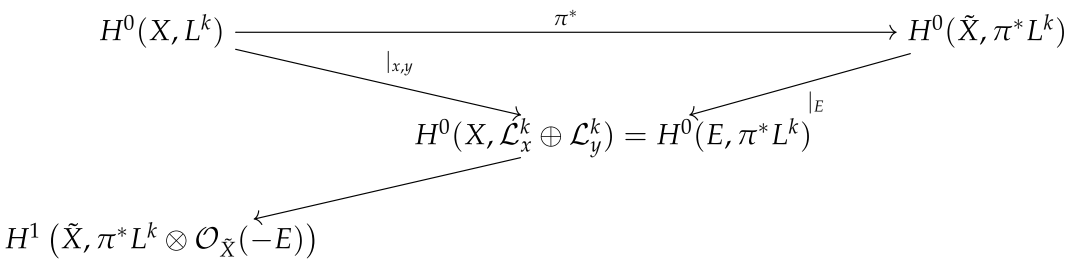
is an embedding, as we discussed in section 1.2, that entails to show that:
(1). The restriction map:

$$H^0(X, L^k) \longrightarrow H^0(X, \mathcal{L}_x^k \oplus \mathcal{L}_y^k)$$

is surjective for all $x \neq y \in X$.
(2). The differential map:

$$d_x : H^0(X, L^k) \longrightarrow H^0(X, \mathcal{L}_x^k \otimes T_x^{*(1,0)} X)$$

is surjective for all $x \in X$.
We'd seen form theorem 2.5, the theorem is right for compact Riemann surfaces, however, for high-dimensional complex manifold, the ideal sheaf $\mathcal{I}_x$ may no longer be invertible, hence the technique lost its power here, fortunately, we can use the blowing up.
Let $\pi : \widetilde{X_{x,y}} \longrightarrow X$ be the blowup of X at two distinct points x, y, the exceptional divisors denoted by $E_x = \pi^{-1}(x), E_y = \pi^{-1}(y)$ with respect to the point x, y, and let $E = E_x + E_y$, we will have the following diagram commutes:



Indeed:
(a). $\pi^* L^k$ is trivial along E_x and E_y, in fact:

$$\begin{aligned} \pi^*|_{E_x} &= \pi^*|_{\pi^{-1}(x)} = \pi^*(L^k|_x) \\ &= \pi^*(\{x\} \times \mathcal{L}_x^k) = E_x \times \mathcal{L}_x^k \end{aligned}$$

Similarly we have $\pi^* L^k|_{E_y} = E_y \times \mathcal{L}_y^k$, hence

$$H^0(X, \mathcal{L}_x^k \oplus \mathcal{L}_y^k) = H^0(E, \pi^* L^k)$$

(b). $\pi^* : H^0(X, L^k) \longrightarrow H^0(\tilde{X}, \pi^* L^k)$ is an isomorphism, in fact, π is a biholomorphism away from E, π^* is injective, by Hartogs theorem [1], any holomorphic section of $\pi^* L^k$ on $\tilde{X} \setminus E_x \cup E_y \cong X \setminus \{x, y\}$ can be extended to a holomorphic section of L^k on the whole X, hence π^* is surjective.
(c). The restriction map $|_E : H^0(\tilde{X}, \pi^* L^k) \longrightarrow H^0(E, \pi^* L^k)$ lies in the long exact sequence induced by the short exact sequence:

$$0 \longrightarrow \pi^* L^k \otimes \mathcal{O}_{\tilde{X}}(-E) \longrightarrow \mathcal{O}_{\tilde{X}}(\pi^* L^k) \longrightarrow \mathcal{O}_E(\pi^* L^k) \longrightarrow 0$$

Hence the the map **(1)** is surjective if and only if $H^1\left(\tilde{X}, \pi^*L^k \otimes \mathcal{O}_{\tilde{X}}(-E)\right) = 0$, in fact, by theorem 3.2 and 3.1, we have

$$\begin{aligned}\pi^*L^k \otimes \mathcal{O}_{\tilde{X}}(-E) &= \pi^*L^k \otimes \mathcal{O}_{\tilde{X}}(-E) \otimes K_{\tilde{X}} \otimes K_{\tilde{X}}^{-1} \\ &= \pi^*L^k \otimes \mathcal{O}_{\tilde{X}}(-E) \otimes K_{\tilde{X}} \otimes \left(\pi^*K_X \otimes \mathcal{O}_{\tilde{X}}((n-1)E)\right)^{-1} \\ &= K_{\tilde{X}} \otimes \left(\pi^*L^{k_1} \otimes \mathcal{O}_{\tilde{X}}(-nE)\right) \otimes \left(\pi^*(L^{k_2} \otimes K_X^{-1})\right)\end{aligned}$$

for some $k > k_1 + k_2$ chosen suitably such that the line bundles $\pi^*L^{k_1} \otimes \mathcal{O}_{\tilde{X}}(-nE)$ and $\pi^*(L^{k_2} \otimes K_X^{-1})$ are positive, then applying Kodaira vanishing theorem (theorem 2.4) we have

$$H^1\left(\tilde{X}, \pi^*L^k \otimes \mathcal{O}_{\tilde{X}}(-E)\right) = 0$$

Similarly for the surjectivity of **(2)**, we can construct the following commutative diagram:

$$\begin{array}{ccc} H^0\left((X, L^k \otimes \mathcal{I}_x)\right) & \xrightarrow{\pi^*} & H^0\left(\tilde{X}, \pi^*L^k \otimes \mathcal{O}_{\tilde{X}}(-E)\right) \\ & \searrow^{d_x} & \downarrow |_E \\ & H^0\left(X, \mathcal{L}_x^k \otimes T_x^{*(1,0)}X\right) = & H^0\left(E, \pi^*L^k \otimes \mathcal{O}_E(-E)\right) \\ & & \downarrow \\ & & H^1\left(\tilde{X}, \pi^*L^k \otimes \mathcal{O}_{\tilde{X}}(-2E)\right) \end{array}$$

Indeed:
(d). Since π^*L^k is trivial along E, we have

$$\begin{aligned}H^0\left(E, \pi^*L^k \otimes \mathcal{O}_{\tilde{X}}(-E)\right) &\cong H^0(E, \pi^*L^k) \otimes H^0(E, \mathcal{O}_E(-E)) \\ &= H^0(X, \mathcal{L}_x^k) \otimes T_x^{*(1,0)}X \\ &= H^0\left(X, \mathcal{L}_x^k \otimes T_x^{*(1,0)}X\right)\end{aligned}$$

(e). The pull back $\pi^* : H^0(X, L^k \otimes \mathcal{I}_x) \longrightarrow H^0(\tilde{X}, \pi^*L^k \otimes \mathcal{O}_{\tilde{X}}(-E))$ is an isomorphism. In fact, holomorphic sections of L^k on X vanishing at x corresponds to the holomorphic sections of π^* vanishing along E.
(f). The restriction map

$$|_E : H^0\left(\tilde{X}, \pi^*L^k \otimes \mathcal{O}_{\tilde{X}}(-E)\right) \longrightarrow H^0\left(E, \pi^*L^k \otimes \mathcal{O}_E(-E)\right)$$

is sited in the long exact sequence induced from the short exact sequence:

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}(-2E) \otimes \pi^*L \longrightarrow \mathcal{O}_{\tilde{X}}(-E) \otimes \pi^*L \longrightarrow \mathcal{O}_E(-E) \otimes \pi^*L \longrightarrow 0$$

Hence the differential map d_x is surjective if and only if $H^1\left(\tilde{X}, \pi^*L^k \otimes \mathcal{O}_{\tilde{X}}(-2E)\right) = 0$, like in (1), we can apply the Kodaira vanishing theorem to deduce it. ■

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