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首都师范大学硕士学位论文

亚纯联络模空间上的辛结构

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Master Thesis

Symplectic Form on Moduli Space of Meromorphic Connections

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摘 要

本文研究了 $\mathbb{P}^1$ 的平凡向量丛上亚纯联络的模空间 $\tilde{\mathcal{M}}^*(\boldsymbol{A})$，我们证明了这个模空间同构于余伴随轨道的辛商，从而继承了一个辛结构。

受到亚纯线性常微分方程的 Stokes 现象的启发，我们还研究了亚纯联络的 Stokes 矩阵，即由联络诱导的一个群胚 $\tilde{\Gamma}$ 的 Stokes 表示，而后我们研究了所有这些表示构成的模空间 $\tilde{M}(\boldsymbol{A})$，并且给出了在 $\mathbb{P}^1$ 上度为 0 的向量丛上的 Riemann-Hilbert 对应，即度为 0 的向量丛上的亚纯联络的模空间 $\tilde{\mathcal{M}}^0(\boldsymbol{A})$ 同构于 Stokes 表示的模空间的一个度为 0 的分支 $\tilde{M}_0(\boldsymbol{A})$. 受 Atiyah 和 Bott 工作的启发，我们利用带有极点的 C^∞ 平坦联络的模空间再次得到了 $\tilde{\mathcal{M}}^0(\boldsymbol{A})$，从而 $\tilde{M}_0(\boldsymbol{A})$ 也继承了一个辛结构。

最后，我们证明了 Riemann-Hilbert 映射是一个辛映射。

关键词: 辛几何，Stokes 矩阵，单值化，亚纯联络

ABSTRACT

We studied the symplectic structure on the moduli space $\tilde{\mathcal{M}}^*(\boldsymbol{A})$ of meromorphic connections on trivial bundles over $\mathbb{P}^1$, and proved it is a symplectic quotient of some coadjoint orbits, hence it inherits a symplectic structure.

Then, inspired by the Stokes phenomenon of meromorphic linear ordinary differential equations, we also investigated the Stokes data of the meromorphic connections, namely, the Stokes representation of a groupoid $\tilde{\Gamma}$, we then studied the moduli space of the Stokes representations, and proved the Riemann-Hilbert correspondence in the case of degree zero bundles, i.e, the moduli space of meromorphic connections on degree zero bundles over $\mathbb{P}^1$, denoted by $\tilde{\mathcal{M}}^0(\boldsymbol{A})$, is isomorphic to the degree zero-component of the moduli space of Stokes representations, denoted by $\tilde{M}_0(\boldsymbol{A})$.

Motivated by Atiyah and Bott's work, we obtained the moduli space $\tilde{\mathcal{M}}^0(\boldsymbol{A})$ from the moduli space of C^∞ flat connections with poles, this gives a symplectic structure on $\tilde{M}_0(\boldsymbol{A})$.

Finally, we showed Riemann-Hilbert correspondence is symplectic.

Key Words: Symplectic Geometry; Monodromy; Stokes Matrices; Meromorphic Connections

Contents

Notation

$\mathbb{A}$	the collection of anti-Stokes directions
$\boldsymbol{A} = ({}^1A^0, ..., {}^mA^0)$	the data of irregular/formal type near each pole a_i
A^0	a diagonal matrix of meromorphic functions on $\mathbb{C}$ with pole only at $z = 0$ and has no holomorphic part
$\tilde{\mathcal{A}}_D(\boldsymbol{A})$	the collection of C^∞ connections with poles on the divisor D and with irregular type $\boldsymbol{A}$
$\tilde{\mathcal{A}}_{\text{fl}}$	the collection of flat C^∞ singular connections
$\text{Aut}(E)$	the bundle automorphisms of E
B_k	the subgroup of G_k with constant equals to identity
$\mathfrak{b}_k$	the Lie algebra of B_k
$\mathbb{C}[\![z]\!]$	the ring of formal power series
$\mathbb{C}\{z\}$	the ring of convergent power series
$D = k_1a_1 + \cdots + k_ma_m$	an effective divisor on $\mathbb{P}^1$
$(df)_p$	the differential map of f at p
$E \longrightarrow \mathbb{P}^1$	a complex vector bundle of rank n on $\mathbb{P}^1$
(E, ∇)	a generic meromorphic connection on E with formal type $\boldsymbol{A}$
$(E, \nabla, \boldsymbol{g})$	a generic meromorphic connection on E with formal type $\boldsymbol{A}$ and compatible framing $\boldsymbol{g}$
$\hat{F} \in G[\![z]\!]$	the formal gauge transformation
$\mathcal{G} = \text{GL}_n\left(C^\infty\left(\mathbb{P}^1\right)\right)$	the gauge transformation group of trivial bundle
$\mathcal{G}_1$	the subgroup of $\mathcal{G}$ containing the elements with taylor expansion equals to identity
$\boldsymbol{g} = ({}^1g_0, ..., {}^mg_0)$	the data of compatible framing near each pole a_i
$\tilde{\Gamma}$	the groupoid induced by some $\boldsymbol{A}$
$G_k = \text{GL}_n\left(\mathbb{C}[\zeta]/\zeta^k\right)$	the $(k-1)-$jet group
$\mathfrak{g}_k$	the Lie algebra of G_k
$\text{GL}_n(\mathbb{C})$	general linear group
$\mathfrak{gl}_n(\mathbb{C})$	the general linear algebra, the Lie algebra of $\text{GL}_n(\mathbb{C})$
$\text{GL}_n(\mathbb{C}[\![z]\!]) = G[\![z]\!]$	the general linear group with entries in $\mathbb{C}[\![z]\!]$
$\text{GL}_n(\mathbb{C}\{z\}) = G\{z\}$	the general linear group with entries in $\mathbb{C}\{z\}$
$\mathcal{H}\left(A^0\right)$	the moduli space of pairs $\left(A, \hat{F}\right)$
L_i	Laurent map defined by taking Laurent expansion at the pole a_i
$\boldsymbol{\Lambda} = ({}^1\Lambda, ..., {}^m\Lambda)$	the data of the exponents of formal monodromy

$\mathcal{M}(\boldsymbol{A})$	the moduli space of all generic meromorphic connections (E,∇) with formal type $\boldsymbol{A}$
$\tilde{\mathcal{M}}(\boldsymbol{A})$	the extended moduli space of all generic meromorphic connections $(E,\nabla,\boldsymbol{g})$ with irregular type $\boldsymbol{A}$
$\mathcal{M}^*(\boldsymbol{A})$	the moduli space of gneric meromorphic connections on trivial bundles with formal type $\boldsymbol{A}$
$\tilde{\mathcal{M}}^*(\boldsymbol{A})$	the extended moduli space of meromorphic connections on trivial bundles with irregular type $\boldsymbol{A}$
$\tilde{\mathcal{M}}^0(\boldsymbol{A})$	the extended moduli space of meromorphic connections on degree 0 bundles with irregular type $\boldsymbol{A}$
$\tilde{M}(\boldsymbol{A})$	the moduli space of Stokes representations of the groupoid determined by $\boldsymbol{A}$
$\tilde{M}_0(\boldsymbol{A})$	the moduli space of degree 0 Stokes representations of the groupoid determined by $\boldsymbol{A}$
$\mathcal{O}$	the sheaf of holomorphic functions
$\mathcal{O}_D$	the sheaf of meromorphic functions with poles on D
O_i	the coadjoint orbit of the jet group G_{k_i} containing ${}^iA^0$
$\tilde{O}_i$	the extended coadjoint orbit
$\Omega(\nabla)$	the curvature of a connection ∇
Ω^r_D	the sheaf of singular smooth $r-$forms with poles on D
Sect_i	the $i-$th Stokes sector
$\widehat{\mathrm{Sect}}_i$	the $i-$th super-Stokes sector
$\mathrm{Sto}_d\left(A^0\right)$	the group of Stokes factors associated to the direction d
$\Sigma_i\left(\hat{F}\right)$	the re-summation (Borel-Laplace) of $\hat{F}$ on some Stokes sector
$\widehat{\mathrm{Syst}}\left(A^0\right)$	the collection of pairs $\left(A,\hat{F}\right)$ where $A=\hat{F}\left[A^0\right]$

0 Introduction

It has been a long history in studying the moduli spaces derived from the Riemann surfaces, for example, the moduli space of complex structures, the moduli space of stable holomorphic vector bundles, the moduli space of representations of the fundamental group of a Riemann surface, etc. All these moduli spaces have complex structures, however, these structures highly depend on the complex structure of the underlying Riemann surface.

Goldman in [Gol84] showed that the symplectic structure on the moduli space of fundamental group representations just depends on the topology of the Riemann surface, which is also called by "the *symplectic nature*" of the representations. So, it is natural to ask, whether other moduli spaces will have the symplectic nature?

In [AB83], Atiyah and Bott found the symplectic nature of flat connections on the principal bundles over a Riemann surface. In this thesis, we will discuss an extended version of Atiyah-Bott's framework—the symplectic nature of meromorphic connections on holomorphic bundles. Although most of the results remain true on an arbitrary compact Riemann surface, most of our works will be discussed over the complex projective line $\mathbb{P}^1$ for convenience.

To be specific, a key result is [Boa99]:

(1). *The moduli space $\mathcal{M}^*(\boldsymbol{A})$ of generic meromorphic connections with formal type $\boldsymbol{A} = \{A_1, ..., A_m\}$ on $\mathbb{P}^1 \times \mathbb{C}^n$ is isomorphic to the symplectic quotient:*

$$\mathcal{M}^*(\boldsymbol{A}) \cong O_1 \times ... \times O_m /\!\!/ GL_n(\mathbb{C})$$

(2) *The extended moduli space $\tilde{\mathcal{M}}^*(\boldsymbol{A})$ of generic meromorphic connections with irregular type $\boldsymbol{A} = \{A_1, ..., A_m\}$ and compatible framing $\boldsymbol{g} = (g_1, ..., g_m)$ on $\mathbb{P}^1 \times \mathbb{C}^n$ is isomorphic to the symplectic quotient:*

$$\tilde{\mathcal{M}}^*(\boldsymbol{A}) \cong \tilde{O}_1 \times ... \times \tilde{O}_m /\!\!/ GL_n(\mathbb{C})$$

As another approach, there is a more topological aspect of meromorphic connections. Recall that a meromorphic connection can be locally regarded as a linear ordinary differential equation, i.e, a meromorphic linear system. According to Riemann-Hilbert correspondence—there is a 1-1 correspondence between the Fuchsian systems and the monodromy representations of the fundamental group. However, for non-Fuchsian systems, that is the order of the poles of the coefficient matrix is higher than 1, the monodromy cannot describe the behavior of the solution of a system completely, there is more deeply phenomenon, called the Stokes phenomenon.

The Stokes phenomenon arises from the multi-summability of a divergent series. To be specific, for linear systems $dy = Ay$, $dy = A^0y$, if A is formally gauge equivalent to A^0, i.e there exists a formal gauge transformation $\hat{F} \in \mathrm{GL}_n(\mathbb{C}[\![z]\!])$ such that

$$\hat{F}[A] = \left(d\hat{F}\right)\hat{F}^{-1} + \hat{F}A\hat{F}^{-1} = A^0$$

then the solutions of A are differed by a left multiplication of $\hat{F}$ to the solutions of A^0. In order to obtain the convergent solutions, one can do the re-summation of those formal solutions, for example, the Borel-Laplace transformation [SM10], however, the new solution obtained by the re-summation process may only converge in some sectorial neighborhoods around the poles of A^0, and on the overlap of two different sectors, the solutions can be very different, they will be differed by a multiplication of a invertible constant matrix, such matrices are called the Stokes factors, they are completely invariants of a linear ODE [VdPS12].

Similar to the monodromy representations of the fundamental group, there is a more general notion, called the Stokes representation of a groupoid, and in this case, the Riemann-Hilbert problem is asking, *does there exists a 1-1 correspondence between the meromorphic linear systems and the Stokes representations of the underlying groupoid?*

In this thesis, we will study the moduli space of the Stokes representations $\tilde{M}(\boldsymbol{A})$. Then, the Riemann-Hilbert correspondence in the case of degree 0 bundles will be given, that is [Boa99]:

The extended moduli space $\tilde{\mathcal{M}}^0(\boldsymbol{A})$ of meromorphic connections on degree 0 bundles with irregular type $\boldsymbol{A}$ is isomorphic to the degree 0 component of $\tilde{M}_0(\boldsymbol{A})$. This isomorphism is called the Riemann-Hilbert map.

As the third approach to meromorphic connections, we will generalise Atiyan-Bott's framework. We will consider the moduli space of C^∞ flat singular connections on C^∞-trivial bundles, denoted by $\tilde{\mathcal{A}}_{\mathrm{fl}}(\boldsymbol{A})/\mathcal{G}_1$, this moduli space is surprisingly isomorphic to $\tilde{\mathcal{M}}^0(\boldsymbol{A})$. Following Atiyah and Bott [AB83], the moduli space $\tilde{\mathcal{A}}_{\mathrm{fl}}(\boldsymbol{A})/\mathcal{G}_1$ is actually the symplectic quotient of the Hamiltonian group action of $\mathcal{G} = \mathrm{GL}_n(C^\infty(\mathbb{P}^1))$ on $\tilde{\mathcal{A}}_D(\boldsymbol{A})$. Hence by Riemann-Hilbert correspondence, $\tilde{\mathcal{M}}^0(\boldsymbol{A})$ and $\tilde{M}_0(\boldsymbol{A})$ both inherit a symplectic structure.

Finally, if we restricted the Riemann-Hilbert map to $\tilde{\mathcal{M}}^*(\boldsymbol{A})$, we will show that this map is in fact symplectic. The key story is involved in the following commutative diagram [Boa99]:

$$\begin{array}{ccccc} & & \tilde{\mathcal{M}}^0(\mathbf{A}) & \xrightarrow{\tilde{\sigma},\cong} & \tilde{\mathcal{A}}_{\mathrm{fl}}(\mathbf{A})/\mathcal{G}_1 \\ & & \uparrow{\scriptstyle i} & & \downarrow{\scriptstyle \tilde{v},\cong} \\ \tilde{O}_1 \times \cdots \times \tilde{O}_m /\!\!/ GL_n(\mathbb{C}) & \xrightarrow{\tilde{\ell},\cong} & \tilde{\mathcal{M}}^*(\mathbf{A}) & \xrightarrow{\tilde{\nu}} & \tilde{M}_0(\mathbf{A}) \end{array}$$

The arrangement of this thesis is as follows:

Chapter 1 will give a quick review of some necessary background materials, including the notions of meromorphic connections, Stokes matrices, Lie groups and their actions, symplectic geometry, etc.

In chapter 2, we will give the symplectic structure on the moduli space of meromorphic connections by establishing the isomorphism between $\tilde{\mathcal{M}}^*(\boldsymbol{A})$ and the symplectic quotient $\tilde{O}_1 \times \cdots \times \tilde{O}_m // \mathrm{GL}_n(\mathbb{C})$, the main results are Theorem 2.1 and Theorem 2.2.

Then, in chapter 3 we will study the Stokes data of the meromorphic connections. First of all, we will introduce the notion of Stokes representations, then, we will define the moduli space of the Stokes representations, $\tilde{M}(\boldsymbol{A})$, and we will show that two connections are in the same equivalent class if and only if they induce the equivalent Stokes representations, this is the one side of Riemann-Hilbert correspondence. At last, we will give an explicit description of $\tilde{M}(\boldsymbol{A})$. The main results are Theorem 3.1 and Theorem 3.2.

Chapter 4 will introduce the third approach to the meromorphic connection. First, we will introduce the basic notion of C^∞ singular connections with poles on D on a C^∞ trivial vector bundle, we will see the moduli space of meromorphic connections on degree 0 bundles is actually isomorphic to the moduli space of flat connections $\tilde{\mathcal{A}}_{\mathrm{fl}}(\boldsymbol{A})/\mathcal{G}_1$, the latter is also isomorphic to the moduli space of degree 0 Stokes representations $\tilde{M}_0(\boldsymbol{A})$, which induces the Riemann-Hilbert correspondence on degree 0 bundles. At last, we will give a symplectic structure on the moduli space $\tilde{\mathcal{A}}_{\mathrm{fl}}(\boldsymbol{A})/\mathcal{G}_1$ by a similar method of Atiyah-Bott, which shall give a symplectic structure on $\tilde{M}_0(\boldsymbol{A})$ as well. The main theorems are Theorem 4.2, Theorem 4.3 and Theorem 4.4.

In chapter 5, we will show the Riemann-Hilbert map is actually symplectic, the main theorem is Theorem 5.1.

1 Background Materials

1.1 Meromorphic Connections

Although the majority of the discussion will be taken place on the trivial bundle over $\mathbb{P}^1$, we still need some general settings. Now, let X be a compact Riemann surface, E is a holomorphic vector bundle with rank n, $D = k_1 a_1 + ... + k_m a_m > 0$ is an effective divisor on X in which each $k_i > 0$ and $a_i \in X$, which will contain the information of poles.

Definition 1.1 (Meromorphic Connection [GH14])**.** A meromorphic connection with poles on the divisor D (each a_i is a pole of order k_i) is a map:

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes K(D)$$

satisfying Leibniz rule: for any $f \in \mathcal{O}_X$ and $s \in \mathcal{E}$, we have

$$\nabla(f \cdot s) = (df) \otimes s + f \nabla s$$

Here $\mathcal{E}$ is sheaf of local sections of E, $K(D)$ is the sheaf of meromorphic 1-forms with prescribed poles on D, $\mathcal{O}_X$ is the sheaf of holomorphic functions on X.

Next, we will give a local depiction of a meromorphic connection.

Remark 1.1. For a local trivialization $\phi_i : E|_{U_i} \longrightarrow U_i \times \mathbb{C}^n$ near each pole a_i, ∇ can be expressed by Laurent expansion

$$\begin{aligned}\nabla &= d - {}^iA \\ &= d - \left(\frac{{}^iA_{k_i}}{z^{k_i}} + \cdots + \frac{{}^iA_1}{z} + {}^iA_0 + \cdots \right) dz\end{aligned}$$

where ${}^iA_j \in \mathrm{End}(\mathbb{C}^n)$. Hence locally, it is a matrix-valued meromorphic 1-form.

The second part in right hand side will sometimes be denoted by $PP_i + HP_i$, which stands for the principal part and holomorphic part in a Laurent expansion respectively. Besides, ${}^iA_{k_i}/z^{k_i} + \cdots + {}^iA_2/z^2$ is called the **irregular part**, ${}^iA_1$ is called the **residue part**, these terminologies arise form the analytical theory of linear ordinary differential equations [BJL79].

Definition 1.2 (Residue of a Meromorphic Connection)**.** In the local trivialization of E near the pole a_i defined above, we define the residue of a meromorphic connection ∇ at the pole a_i to be the trace of ${}^iA$:

$$\mathrm{Res}_{a_i}(\nabla) := \mathrm{Tr}\left({}^iA\right)$$

We define the residue of ∇ to be the sum of all residues at each pole a_i:

$$\operatorname{Res}(\nabla) := \sum_{i=1}^{m} \operatorname{Tr}\left({}^{i}A\right)$$

Remark 1.2. It is not hard to see that the trace of ${}^{i}A$ doesn't depend on the choices of local trivialization. Indeed, if we choose another trivialization $\psi_i : E|_{V_i} \longrightarrow V_i \times \mathbb{C}^n$, where $a_i \in V_i \cap U_i$, then the map

$$g_i(z) = \phi_i \circ \psi_i^{-1} : U_i \cap V_i \longrightarrow GL_n(\mathbb{C})$$

defines a matrix-valued holomorphic function, and the expressions in different local trivialization will be differed by a gauge transformation:

$${}^{i}A' = g_i(z)\,{}^{i}A g_i^{-1}(z) + (dg_i)g_i^{-1}$$

Since the first part of right hand side doesn't change the eigenvalues and diagonalisability, the second part is holomorphic which will never impact on the principal part, this definition is well-defined. ■

The meromorphic connections are not permitted to be appearing arbitrarily, it was controlled by the properties of the vector bundle E, a theorem of Pereira [Per22] asserts that, the Chern class determined by the residue divisor of a flat meromorphic connection ∇ on a line bundle L over a compact complex manifold X must equal to the negative first Chern class of the line bundle:

$$c(\operatorname{Res}(\nabla)) = -c_1(L) \in H^2(X, \mathbb{C})$$

For low-dimensional case, especially for $X = \mathbb{P}^1$, ∇ is identically flat, and the proof will be easier.

Lemma 1.1. Let ∇ be a meromorphic connection on a holomorphic line bundle L over $\mathbb{P}^{\mathbb{1}}$, which has poles on the divisor $D = \sum_i k_i a_i$, then residue of ∇ equals to the negative degree of the line bundle L:

$$\deg L = -\operatorname{Res}(\nabla)$$

proof. Let

$$U_1 = \left\{[x, y] \in \mathbb{P}^1 | x \neq 0\right\} \quad U_2 = \left\{[x, y] \in \mathbb{P}^1 | y \neq 0\right\}$$

be the standard open cover on $\mathbb{P}^1$, the transition function of L is given by [For12]

$$g_{12}[x, y] = \left(\frac{x}{y}\right)^{\deg L} : U_1 \cap U_2 \longrightarrow \mathbb{C}^*$$

Without the loss of the generality, we can assume all poles of ∇ lie in U_1, i.e, $a_1, .., a_m \in U_1$.

Now, we choose a nowhere vanishing section of L on U_1, namely $\sigma_1 \in \Gamma(U_1; L)$, and σ_2 a section of L on U_2 determined by

$$\sigma_2 = g_{12}\sigma_1 : U_2 \longrightarrow L \qquad (*)$$

hence no hard to see σ_2 is also nowhere vanishing on U_2, thus we can find two forms $\eta_1, \eta_2 \in \mathcal{K}(D)$ such that

$$\nabla\sigma_1 = \eta_1 \otimes \sigma_1 \quad \nabla\sigma_2 = \eta_2 \otimes \sigma_2$$

Hence by $(*)$, we have

$$\frac{dg_{12}}{g_{12}} = \eta_2 - \eta_1$$

As we assuming U_1 containing all poles of ∇, hence η_2 is in fact a holomorphic 1-form, and η_1 has the same residue as ∇, i.e, $\text{Res}\eta_1 = \text{Res}\nabla$.

Now, we use the coordinate $z = x/y$ on $U_1 \cap U_2$, and consider the equality

$$\eta_1 = \eta_2 - \frac{\deg L}{z}$$

by taking residues on both sides yields:

$$\text{Res}(\nabla) = \text{Res}(\eta_1) = -\deg L \qquad \blacksquare$$

Remark 1.3. This result remains true for a rank $n \geqslant 2$ vector bundle E over $\mathbb{P}^1$, in fact, we can consider the induced connection on the determinant line bundle $\bigwedge^n E$, and observe that the residue will not be changed.

Next, we will define a kind of meromorphic connections with a good property, namely *generic*, which will play a vital role in the rest of this thesis.

Definition 1.3 (generic connection [Boa99])**.** A meromorphic connection ∇ is generic, if at each a_i, the leading coefficient ${}^iA_{k_i}$ is diagonalisable and the eigenvalues are

$$\begin{cases} \text{distinct} & k_i \geqslant 2 \\ \text{distinct} \pmod{\mathbb{Z}} & k_i = 1 \end{cases}$$

Remark 1.4. Again, this definition is independent of the choice of local trivialisation (cf. Remark 1.2) hence well-defined.

Now, since the generic connection is of our interests, the leading coefficient is always diagonalisable and having enough distinct eigenvalues, we need to know how to "change" a generic connection into the one with diagonal leading coefficient:

Definition 1.4 (Compatible Framing [Boa99]). A compatible framing at the pole a_i associated to a generic connection ∇ is an isomorphism

$$ {}^i g : E_{a_i} \xrightarrow{\cong} \mathbb{C}^n $$

such that the leading coefficient of ∇ is diagonal along any local trivialization which extends ${}^i g$.

Again, locally, it means that there is an ${}^i g \in GL_n(\mathbb{C})$ such that ${}^i g A_{k_i} {}^i g^{-1}$ is diagonal.

It is natural to give an "idol" among those generic meromorphic connections, this idol shall have a very nice form—it is diagonal for every term.

Definition 1.5 (Nice Formal Form [Boa99]). At each pole a_i, a nice formal form, which denoted by $d - {}^i A^0$, is a meromorphic connection together with a local trivialization such that the local expression is diagonal and has no holomorphic part:

$$ \begin{aligned} {}^i A^0 &= \left(\frac{{}^i \Lambda^0_{k_i}}{z^{k_i}} + \cdots + \frac{{}^i \Lambda^0}{z} \right) dz \\ &:= d({}^i Q) + {}^i \Lambda^0 \frac{dz}{z} \end{aligned} $$

where ${}^i Q$ is a diagonal matrix of meromorphic functions and ${}^i \Lambda^0$ is a diagonal constant matrix.

A connection ∇ with compatible framing ${}^i g$ at a_i is said to have **irregular type** ${}^i A^0$ if ${}^i g$ extends to a formal trivialization near a_i in which the difference of ∇ and $d - {}^i A^0$ is a matrix of 1-forms with just simple pole.

Remark 1.5. Again, we may assume $\nabla = d - A$ in some local trivialization near a_i, it has a irregular type $d - {}^i A^0$ implies there exists a $g(z) \in GL_n(\mathbb{C}[\![z]\!])$ with $g(a_i) = {}^i g$ such that

$$ gAg^{-1} + (dg)g^{-1} = d({}^i Q) + {}^i \Lambda \frac{dz}{z} $$

these ${}^i \Lambda$ are called the **exponents of formal monodromy**.

Now, we denoted by $\boldsymbol{A} = ({}^1 A^0, ..., {}^m A^0)$ the data of irregular types near each pole a_i, and $\boldsymbol{g} = ({}^1 g, ..., {}^m g)$ the data of compatible framing.

The moduli space we will concern is defined as follows

Definition 1.6 (Moduli Space [Boa99]). (1) The moduli space of meromorphic connections on E, denoted by $\mathcal{M}_E(\boldsymbol{A})$, is the set of gauge isomorphism classes of all generic connections on E which are formal equivalent to $\boldsymbol{A}$ near each pole a_i.

$$ \mathcal{M}_E(\boldsymbol{A}) = \{\nabla \,\text{generic} : \text{formal equivalent to}\, \boldsymbol{A}\} / \mathrm{Aut}(E) $$

(2) Similarly, the extended moduli space $\tilde{\mathcal{M}}_E(\boldsymbol{A})$ is defined as the set of gauge isomorphism classes of all generic connections on E with irregular type $\boldsymbol{A}$ near each pole a_i:

$$\tilde{\mathcal{M}}_E(\boldsymbol{A}) = \{(\nabla, \boldsymbol{g})\,\text{generic}:\ \text{has irregular type}\,\boldsymbol{A}\,\text{via}\,\boldsymbol{g}\}/\text{Aut}(E)$$

Here $\text{Aut}(E)$ is the gauge transformation group of E (bundle automorphisms).

Remark 1.6. (1). In the rest part of this thesis, the most interesting case will be $E = \mathbb{P}^1 \times \mathbb{C}^n$, and those moduli spaces will be denoted by $\mathcal{M}^*(\boldsymbol{A})$ and $\tilde{\mathcal{M}}^*(\boldsymbol{A})$ respectively.

(2). The collection of all equivalent classes of pairs (E, ∇) will be denoted by $\mathcal{M}(\boldsymbol{A})$, and for pairs $(E, \nabla, \boldsymbol{g})$ will be denoted by $\tilde{\mathcal{M}}(\boldsymbol{A})$.

(3). By Lemma 1.1, the moduli space $\mathcal{M}_E(\boldsymbol{A})$ is non-empty unless

$$\sum_{i=1}^{m} \text{Tr}\left({}^i\Lambda^0\right) = -\deg E$$

where ${}^i\Lambda$ is the residue part of the nice formal form ${}^iA^0$ appeared in Definition 1.5, and the extended moduli space $\tilde{\mathcal{M}}_E(\boldsymbol{A})$ is non-empty unless

$$\sum_{i=1}^{m} \text{Tr}\left({}^i\Lambda\right) = -\deg E$$

where ${}^i\Lambda$ is the exponents of formal monodromy of ∇, which appeared in Remark 1.5.

1.2 Stokes Matrices of Linear Ordinary Differential Equations

A very important case in ordinary differential equations is the linear case on the complex plane:

$$d\boldsymbol{y} = A(z)\boldsymbol{y}$$

where $A(z)$ is a matrix-valued meromorphic 1-form on $\mathbb{C}$ with poles $a_1, ..., a_m$.

However, it is more interesting to let the linear ordinary differential equations be defined at ∞, hence then, this differential equation is defined on the complex projective line $\mathbb{P}^1$, in this sense, the study of meromorphic connections on $\mathbb{P}^1$ is equivalent to study the local theory of linear ODEs.

Different from the nonlinear equations, the solutions of a linear one will only have singularities near the poles of $A(z)$, the behavior of the solutions near these poles are very interesting. A very significant invariant will be appeared near these poles, called the family of Stokes matrices, it will be introduced in this section.

1.2.1 Stokes Matrices

We will assume our linear ODEs are generic (cf. Definition 1.3) and with only one pole at $z = 0$. Let $dy = A^0 y$ be a diagonal generic meromorphic linear ODE, we can write A^0 as

$$A^0 = dQ + \Lambda^0 \frac{dz}{z}$$

here Q is a diagonal matrix of meromorphic functions, Λ^0 is a diagonal constant matrix. Now, write the entries of Q in terms of Laurent expansions:

$$Q = \begin{pmatrix} q_1 & & & \\ & q_2 & & \\ & & \ddots & \\ & & & q_n \end{pmatrix}$$

where $q_i \in \mathbb{C}\{z\}[1/z]$, define q_{ij} to be the leading term of $q_i - q_j$, for example, if we write

$$q_i - q_j = \frac{a}{z^{k-1}} + \frac{b}{z^{k-2}} + \cdots$$

then $q_{ij} = a/z^{k-1}$.

Definition 1.7 (Anti-Stokes Directions [Boa99])**.** The anti-Stokes directions $\mathbb{A} \subset \mathbb{S}^1$ are the directions $d \in \mathbb{S}^1$ such that $q_{ij}(z) \in \mathbb{R}_{<0}$ for all z on the ray specified by d, for some ij.

Let d be an anti-Stokes direction, we define the following data of this direction:

(1). The **roots** of d are the ordered pair (ij) *supporting* d:

$$\text{Roots}(d) = \{(ij) : q_{ij}(z) \in \mathbb{R}_{<0}, z \in \mathbb{R}_{>0} e^{id}\}$$

(2). The **multiplicity** $\text{Mult}(d)$ is the cardinality of $\text{Roots}(d)$.

(3). The group of **Stokes factors** associated to d is the group

$$\text{Sto}_d(A^0) = \{K \in \text{GL}_n(\mathbb{C}) : K_{ij} = \delta_{ij} \text{ unless } (ij) \in \text{Roots}(d)\}$$

Remark 1.7. It is not hard to find the following facts about the anti-Stokes directions:

(1). The Stokes group $\text{Sto}_d(A^0)$ is a unipotent subgroup of $\text{GL}_n(\mathbb{C})$

(2). The anti-Stokes directions $\mathbb{A}$ have $\pi/(k-1)$ rotational symmetry, here k is the order of the pole 0, i.e, if $q_{ij}(z) \in \mathbb{R}_{<0}$, then

$$q_{ij}\left(z \cdot e^{\frac{\pi\sqrt{-1}}{k-1}}\right) \in \mathbb{R}_{<0}$$

hence the number of all anti-Stokes directions $r = |\mathbb{A}|$ is divisible by $2k-2$, we denote $\ell = r/2(k-1)$.

By the last item in the Remark 1.7, we can refer to an $\ell-$tuple

$$\boldsymbol{d} = (d_1, ..., d_\ell) \subset \mathbb{A}$$

of consecutive anti-Stokes directions as a *half-period*, the half-period will define an order of the set $\{q_1, ..., q_n\}$:

$$q_i < q_j \quad \Leftrightarrow \quad (ij) \text{ is a root of } d \in \boldsymbol{d}$$

Lemma 1.2 ([Boa99]). For each half-period $\boldsymbol{d}$, there exists a permutation matrix P which can upper/lower-triangularize all matrices in the group $\mathrm{Sto}_d(A^0)$ for any $d \in \boldsymbol{d}$.

proof. Indeed, define π to be a permutation of the set $\{1, ..., n\}$:

$$\pi(i) < \pi(j) \quad \Leftrightarrow \quad q_i < q_j \quad \Leftrightarrow \quad (ij) \text{ is a root of } d \in \boldsymbol{d}$$

define $P = (P)_{ij}$, where $(P)_{ij} = \delta_{\pi(i)j}$, this is the desired permutation matrix. ∎

Moreover, we have a much stronger statement:

Lemma 1.3 ([BJL79]). (1). Let $\boldsymbol{d} = (d_1, ..., d_\ell)$ be a half-period, we have the isomorphism:

$$\prod_{d \in \boldsymbol{d}} \mathrm{Sto}_d(A^0) \cong PU_{\pm}P^{-1}$$
$$(K_1, ..., K_\ell) \mapsto K_\ell \cdots K_2 K_1$$

where $U_\pm$ is the upper/lower-triangulated subgroup of $\mathrm{GL}_n(\mathbb{C})$.

(2). If we label the rest of $\mathbb{A}$ as $d_{\ell+1}, ..., d_r$ in the positive order (anti-clockwise), we have the isomorphism:

$$\prod_{d \in \mathbb{A}} \mathrm{Sto}_d(A^0) \cong (U_+ \times U_-)^{k-1}$$
$$(K_1, ..., K_r) \mapsto (S_1, ..., S_{2k-2})$$

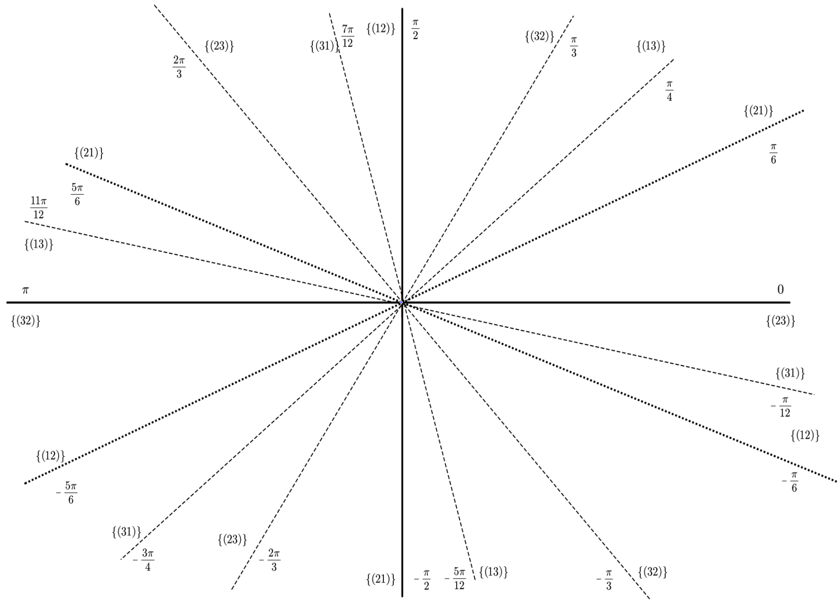
here $S_i = P^{-1} K_{i\ell} \cdots K_{(i-1)\ell+1} P$.

Example: We assume

$$A^0 = d \begin{pmatrix} \frac{i}{z^3} & & \\ & \frac{1}{z^2} & \\ & & \frac{1}{z^3} \end{pmatrix}$$

then $q_{12} = \sqrt{-1}/z^3$, $q_{13} = (\sqrt{-1} - 1)/z^3$, $q_{23} = -1/z^3$, also $q_{ij} = -q_{ji}$, hence we can determine all anti-Stokes directions in following picture:

All these 18 directions are of multiplicity 1, the Stokes group for the direction, $d = \pi/4$



The anti-Stokes directions of A^0

for instance, is the collection of all invertible matrices with the form

$$\mathrm{Sto}_{\frac{\pi}{4}}(A^0) = \left\{ \begin{pmatrix} 1 & & * \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}) \middle| * \in \mathbb{C} \right\}$$

Choose a half-period, namely $\boldsymbol{d} = \{0, \pi/6, \pi/4\}$, the permutation matrix associated to this half-period is

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

all matrices in $\prod_{d \in \boldsymbol{d}} \mathrm{Sto}_d(A^0)$ can be upper-triangularized via P.

1.2.2 The Local Moduli Space $\mathcal{H}\left(A^0\right)$

Next, we shall focus on the Stokes matrices for the linear ODE that with formal type A^0. We denote $\mathrm{Syst}\left(A^0\right)$ the set of all linear ODEs with formal form A^0, i.e:

$$\mathrm{Syst}\left(A^0\right) = \left\{ d - A \middle| A = \hat{F}\left[A^0\right], \hat{F} \in \mathrm{GL}_n(\mathbb{C}[\![z]\!]) \right\}$$

here $\hat{F}[A^0]$ represents for the gauge transformation on A^0:

$$\hat{F}\left[A^0\right] := \left(d\hat{F}\right)\hat{F}^{-1} + \hat{F}A^0\hat{F}^{-1}$$

Then, we will define another set which is larger than $\mathrm{Syst}(A^0)$ (by considering the data of formal gauge transformation $\hat{F}$ as well):

Definition 1.8 ([Boa99]). Define

$$\widehat{\mathrm{Syst}}\left(A^0\right) = \left\{\left(A, \hat{F}\right) | A \in \mathrm{Syst}\left(A^0\right), \hat{F} \in \mathrm{GL}_n(\mathbb{C}[\![z]\!]), A = \hat{F}\left[A^0\right]\right\}$$

an element $\left(A, \hat{F}\right) \in \widehat{\mathrm{Syst}}\left(A^0\right)$ is called a *marked pair.*

Let $G\{z\} = \mathrm{GL}_n(\mathbb{C}\{z\})$ acts by gauge transformation on $\widehat{\mathrm{Syst}}(A^0)$, define:

$$\mathcal{H}(A^0) = \widehat{\mathrm{Syst}}(A^0) \Big/ G\{z\}$$

This $\mathcal{H}(A^0)$ is the local version of the extended moduli space $\tilde{\mathcal{M}}(\boldsymbol{A})$, and an element in $\mathcal{H}(A^0)$ is a gauge equivalent class of linear ODEs which are formal equivalent to A^0.

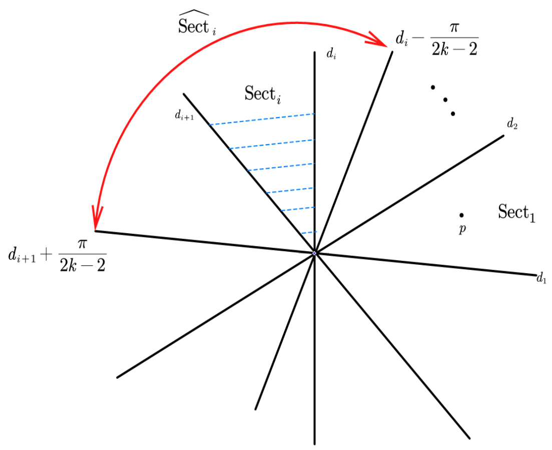
Next, we define a labelling convention of the anti-Stokes directions of A^0. First, we fixed a point p in one of the Stokes sectors bounded by two consecutive anti-Stokes rays, label the first anti-Stokes ray when turning in a positive sense (anti-clockwise) from p as d_1, and label the rests as $d_2, ..., d_r$, denote

$$\mathrm{Sect}_i := \mathrm{Sect}(d_i, d_{i+1})$$

to be the $i-$th sector, and

$$\widehat{\mathrm{Sect}}_i = \mathrm{Sect}\left(d_i - \frac{\pi}{2k-2}, d_{i+1} + \frac{\pi}{2k-2}\right)$$

the i-th supersector.



labelling convention

Lemma 1.4 ([Boa99]). If $\hat{F} \in \mathrm{GL}_n(\mathbb{C}[\![z]\!])$ is a formal transformation such that $A = \hat{F}\left[A^0\right]$

has convergent entries. Set the radius of the sectors Sect_i and $\widehat{\mathrm{Sect}}_i$ to be less than the radius of convergence of A, then the following statements hold:

(1). The formal transformation can be replaced by a convergent one, i.e, one can determine an invertible matrix of holomorphic functions (in a canonical way) $\Sigma_i(\hat{F}) \in \mathrm{GL}_n(\mathcal{O}(\mathrm{Sect}_i))$ on each sector Sect_i such that

$$\Sigma_i(\hat{F})\left[A^0\right] = A$$

(2). $\Sigma_i\left(\hat{F}\right)$ can be analytically continued to the super-sector $\widehat{\mathrm{Sect}}_i$.

(3). For $g \in \mathrm{G}\{z\}$ and $t \in \mathbb{T}$ (the toric group $\mathbb{T}$ can be viewed as the diagonal subgroup of $\mathrm{GL}_n(\mathbb{C})$), we have

$$\Sigma_i\left(g\hat{F}t^{-1}\right) = g\Sigma_i\left(\hat{F}\right)t^{-1}$$

The canonical method mentioned in the last lemma is called the Borel-Laplace transformation [SM10].

Now, choose a branch of $\log z$ along d_1 and extend it in a positive sense across other sectors. Recall that, if we write $A^0 = dQ + \Lambda^0 dz/z$, then the solution of the differential equation $dy = A^0 y$ can be write as $z^{\Lambda^0}e^Q$ (in the given branch), and for any $A = \hat{F}[A^0]$, the solution of $dy = Ay$ can be write as

$$Y_i = \Sigma_i\left(\hat{F}\right)\left[A^0\right] z^{\Lambda^0} e^Q$$

on each sector Sect_i, $i = 1, ..., r$. We denote

$$\kappa_i := \Sigma_i\left(\hat{F}\right)^{-1} \cdot \Sigma_{i-1}\left(\hat{F}\right) \in \mathrm{GL}_n\left(\mathcal{O}\left(\widehat{\mathrm{Sect}}_i \cap \widehat{\mathrm{Sect}}_{i-1}\right)\right)$$

Remark 1.8. κ_i asymptotic to 1 at 0 in the sector $\widehat{\mathrm{Sect}}_i \cap \widehat{\mathrm{Sect}}_{i-1}$, moreover:

$$\kappa_i\left[A^0\right] = A^0$$

Definition 1.9. The **Stokes factors** of a linear ODE $(A, \hat{F}) \in \widehat{\mathrm{Syst}}(A^0)$ are

$$K_i := e^{-Q} z^{-\Lambda^0} \cdot \kappa_i \cdot z^{\Lambda^0} e^Q \quad i = 1, ..., r$$

Lemma 1.5 ([Boa99] [VdPS12]). (1). The Stokes factor K_i is constant and lies in the group $\mathrm{Sto}_{d_i}(A^0)$ for each i.

(2). Stokes factors are the complete invariants of a linear ODE, i.e, $(A, \hat{F})$ and $(A', \hat{F}')$ are in the same equivalent class in $\mathcal{H}(A^0)$ if and only if they have the same Stokes factors.

proof. (1). To be simplified, we write $Y_0 = z^{\Lambda^0}e^Q$, which is the fundamental solution of $dy = A^0 y$, hence by remark 1.5, we have

$$\begin{aligned} dK_i &= d\left(Y_0^{-1}\right)\cdot \kappa_i \cdot Y_0 + Y_0^{-1}\cdot d\kappa_i \cdot Y_0 + Y_0^{-1}\cdot \kappa_i \cdot dY_0 \\ &= -A^0 Y_0^{-1}\kappa_i Y_0 + Y_0^{-1}\kappa_i A^0 Y_0 + Y_0^{-1} d\kappa_i Y_0 \\ &= -A^0 Y_0^{-1}\kappa_i Y_0 + Y_0^{-1}\kappa_i A^0 Y_0 + Y_0^{-1}\left(A^0\kappa_i - \kappa_i A^0\right) Y_0 \\ &= \left(Y_0^{-1}A^0 - A^0 Y_0^{-1}\right)\kappa_i Y_0 \\ &= 0 \end{aligned}$$

the last step comes from the fact that A^0 and Y_0 are diagonal matrices.

Hence K_i is a constant matrix. To see $K_i \in \mathrm{Sto}_{d_i}(A^0)$, by Remark 1.8, we know

$$e^{-Q} K_i e^{Q} = z^{-\Lambda^0}\kappa_i z^{\Lambda^0} \longrightarrow I_n$$

as $z \to 0$ within the intersection $\widehat{\mathrm{Sect}}_i \cap \widehat{\mathrm{Sect}}_{i-1}$, it forces the $(m,n)-$th entry of K_i to be

$$(K_i)_{mn} = \delta_{mn}$$

unless $e^{q_m - q_n} \longrightarrow 0$ as $z \to 0$, this is equivalent to $K_i \in \mathrm{Sto}_{d_i}(A^0)$.

(2) follows directly from the 3rd statement in Lemma 1.4. ■

Remark 1.9. By Lemma 1.2, for each Stokes group $\mathrm{Sto}_{d_i}(A^0)$, there exists a permutation matrix P_i such that all matrices in that group can be upper/lower-triangularized by P_i, we will call the upper/lower-triangular matrices

$$S_i := P_i K_i P_i^{-1}$$

the **Stokes Matrices** of the ODE $d - A = 0$.

Combining with Lemma 1.2 and Lemma 1.4, we have

Lemma 1.6 ([BJL79]). We have the isomorphism:

$$\mathcal{H}\left(A^0\right) \cong \left(U_+ \times U_-\right)^{k-1}$$

$$\left[A, \hat{F}\right] \longrightarrow \left(S_1, ..., S_{2k-2}\right)$$

where $S_i = P^{-1} K_{i\ell} \cdots K_{(i-1)\ell+1} P$, that is the product of all Stokes factors of $\left[A, \hat{F}\right] \in \mathcal{H}\left(A^0\right)$ in the $i-$th half-period in a converse order.

Remark 1.10. In the Galois theory of linear ODEs (i.e, the differential Galois theory), the Stokes factors are the significant elements in the differential Galois group $\mathrm{Gal}\left(A, \hat{F}\right)$ of $\left[A, \hat{F}\right]$, in fact, this group is a linear algebraic group generated by all Stokes matrices together with the differential Galois group of $\mathrm{Gal}(A^0)$ [VdPS12], that's why we also call them the "complete invariants" of a linear ODE.

1.3 Lie Groups Actions

In this section, we will introduce some basic notions about Lie group actions on manifolds. In general, groups are used to describe the symmetries of some objects, for example, the dihedral group D_n, we can use the groups actions on a set to investigate the symmetries of that set (and conversely, we can use one group acts on various sets to recover the structure of the group, that is the idea of representation theory), sometimes, the symmetries of a set is not discrete, the circle $\mathbb{S}^1$ for instance, it has *continuously* symmetries, so we need to use a kind of groups which admitted a "continuity" structure to characterise such symmetries, that is the *Lie Groups*.

Definition 1.10 (Lie groups actions [Aud04]). Let G be a Lie group, M a smooth manifold, a $G-$action on M is a group morphism

$$G \longrightarrow \mathrm{Diff}(M)$$

where $\mathrm{Diff}(M)$ is the diffeomorphism group of M.

An action of G on M will be denoted by

$$G \times M \longrightarrow M, \qquad (g, x) \mapsto g \cdot x$$

Example 1. The Lie group G itself is a manifold, and G acts itself by left multiplication:

$$G \times G \longrightarrow G, \qquad (g, h) \mapsto L_g h := gh$$

notice that the tangent map is actually an isomorphism between

$$(dL_{g^{-1}})_g : T_g G \longrightarrow \mathfrak{g}$$

hence

Lemma 1.7. The tangent bundle TG and cotangent bundle of a Lie group are both trivial:

$$TG \cong G \times \mathfrak{g} \qquad T^*G \cong G \times \mathfrak{g}^*$$

Example 2. G can also act on itself by adjoint action:

$$G \times G \longrightarrow G, \qquad (g, h) \mapsto \mathrm{ad}_g h := ghg^{-1}$$

notice that the differential map is an automorphism of the Lie algebra $\mathfrak{g}$ of G:

$$\mathrm{Ad}_g := (d\,\mathrm{ad}_g)_e : \mathfrak{g} \longrightarrow \mathfrak{g}$$

we call the map

$$\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g}), \qquad g \mapsto (d\,\mathrm{ad}_g)_e$$

the *adjoint representation* of G on $\mathrm{GL}(\mathfrak{g})$. Similarly, for every $g \in G$, the adjoint action also induces a representation of G on the dual Lie algebra $\mathfrak{g}^*$ by the cotangent map:

$$\mathrm{Ad}^* : G \longrightarrow \mathrm{GL}(\mathfrak{g}^*), \qquad g \mapsto \mathrm{Ad}^*_g := (d\,\mathrm{ad}_g)^*_e$$

where $g \mapsto (d\,\mathrm{ad}_g)^*_e$ is the cotangent map of ad_g, this representation is called the *coadjoint action of G on $\mathfrak{g}^*$*.

If two elements in G are very "close", then we can "differentiate" this action, that yields the notion of fundamental vector field.

Definition 1.11 (Fundamental Vector Field [Aud04])**.** If G acts on M, for any $X \in \mathfrak{g}$, the vector field defined by

$$\underline{X}(x) := \left.\frac{d}{dt}\right|_{t=0} (\exp tX) \cdot x$$

is called the *fundamental vector field of X*, where $\exp : \mathfrak{g} \longrightarrow G$ is the exponential map.

Lemma 1.8 ([Aud04])**.** The fundamental vector field of $X \in \mathfrak{g}$ associated to the adjoint action on $\mathfrak{g}$ is

$$\underline{X}(Y) = [X, Y]$$

and the one associated to the coadjoint action is

$$\langle \underline{X}(\xi), Y \rangle = \langle \xi, [Y, X] \rangle$$

where $X, Y \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$, $\langle \cdot, \cdot \rangle$ is paring

$$\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \longrightarrow \mathbb{R}$$

proof. First, for adjoint action $\mathfrak{g}$, let $X, Y \in \mathfrak{g}$, by definition, we have

$$\begin{aligned}
\underline{X}(Y) &= \left.\frac{d}{dt}\right|_{t=0} (d\mathrm{ad}_{\exp tX})_e Y \\
&= \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} (\exp tX) \exp sY (\exp -tX) \\
&= [X, Y]
\end{aligned}$$

for the case of coadjoint action on $\mathfrak{g}^*$, for any $\xi \in \mathfrak{g}^*$, observe that

$$\begin{aligned} \langle \underline{X}(\xi), Y \rangle &= \frac{d}{dt}\bigg|_{t=0} \langle \mathrm{Ad}^*_{\exp tX}\xi, Y \rangle \\ &= \frac{d}{dt}\bigg|_{t=0} \langle \xi, \mathrm{Ad}_{-\exp tX} Y \rangle \\ &= \langle \xi, [Y, X] \rangle \qquad \blacksquare \end{aligned}$$

In practice, we call the map

$$\mathfrak{g} \times M \longrightarrow TM, \qquad (X, x) \mapsto \underline{X}(x)$$

the *infinitesimal action* of G. When G is a connected compact Lie group, we have a good correspondence property between Lie groups and Lie algebras, and some properties of $G-$actions can be reduced to the properties of its infinitesimal actions.

1.4 Symplectic Geometry and Hamiltonian Lie Groups Actions

Definition 1.12 (Symplectic Manifolds [Aud04])**.** A differential manifold M is called symplectic, if there exists a smooth non-degenerate closed $2-$form ω:

$$\omega_x : T_xM \times T_xM \longrightarrow \mathbb{R}$$

Here are several significant examples which will play an important role in this thesis.

Example 1 (cotangent bundle) For any smooth manifold M, its cotangent bundle T^*M has a natural symplectic structure, called the *canonical symplectic form*:
Suppose $\pi : T^*M \longrightarrow M$ is the projection of the cotangent bundle, for $p = (x, \xi) \in T^*M$, we can define its cotangent map:

$$(d\pi)^*_p : T^*_xM \longrightarrow T^*_p(T^*M)$$

hence

$$\alpha_p := (d\pi)^*_p\xi = \xi \circ (d\pi)_p$$

defined a $1-$form on T^*M, it is called the *Liouville 1-form* or *tautological 1-form*, then the formula

$$\omega = d\alpha$$

defined a symplectic form on T^*M. If we choose a coordinate chart $U \subset M$, the coordinate on T^*U denoted by $x_1, ..., x_n, \xi_1, ..., \xi_n$, then the Liouville 1-form can be locally expressed by

$$\alpha = \sum_{i=1}^{n} \xi_i dx_i$$

hence the canonical symplectic form is

$$\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i$$

There is a special case for the cotangent bundle, that is the cotangent bundle of a Lie group G, by Lemma 1.7, T^*G is trivial, and it is important to write down this canonical symplectic form in terms of the operations on Lie groups and Lie algebras.

Lemma 1.9 ([Boa99]). The canonical symplectic form on $T^*G \cong G \times \mathfrak{g}^*$ at the point $(g, \xi) \in T^*G$ is given by

$$\omega_{(g,\xi)}((X, \phi), (Y, \psi)) = \langle \phi, Y \rangle - \langle \psi, X \rangle - \langle \xi, [X, Y] \rangle$$

proof. Recall that, if $\alpha \in \Omega^1(M)$ is a 1-form on M, then for any vector field $X : M \longrightarrow TM$, $\alpha(X)$ is a sooth function on M, and the exterior differential of α is defined by

$$(d\alpha)(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$$

now, we use the left-trivialization to identify the tangent space $T_{(g,\xi)}T^*G$ with $\mathfrak{g} \times \mathfrak{g}^*$, choose $(X, \phi) \in \mathfrak{g} \times \mathfrak{g}^*$, and the Liouville 1-form on T^*G is given by

$$\alpha_{(g,\xi)}(X, \phi) = \langle \xi, X \rangle$$

now for another $(Y, \psi) \in \mathfrak{g} \times \mathfrak{g}^*$, we have

$$\begin{aligned} \omega_{(g,\xi)}((X, \phi), (Y, \psi)) &= (d\alpha)((X, \phi), (Y, \psi)) \\ &= (X, \phi)_{(g,\xi)}(\alpha(Y, \psi)) - (Y, \psi)_{(g,\xi)}(\alpha(X, \phi)) - \alpha([(X, \phi), (Y, \psi)]) \\ &= \langle \phi, Y \rangle - \langle \psi, X \rangle - \langle \xi, [X, Y] \rangle \qquad \blacksquare \end{aligned}$$

Definition 1.13 (Hamiltonian Action, Moment Map [Aud04]). A Lie group $G-$action on a symplectic manifold M is called Hamiltonian, if there exists a map

$$\mu : M \longrightarrow \mathfrak{g}^*$$

such that for any $X \in \mathfrak{g}$, the function defined by

$$\mu_X(x) := \langle \mu(x), X \rangle : M \longrightarrow \mathbb{R}$$

satisfies the condition

$$\iota_{\underline{X}}\omega = -d\mu_X$$

Lemma 1.10 (Symplectic Reduction [Aud04])**.** If $0 \in \mathfrak{g}^*$ is a regular value of the moment map μ, then there is a symplectic structure on the quotient space $\mu^{-1}(0)/G$, this space is called the *symplectic reduction* of M, this quotient is called the *symplectic quotient*, denoted by $M/\!\!/G$

2 Meromorphic Connections on Trivial Bundles over $\mathbb{P}^1$

In this section we will give and prove the existence of the symplectic structure on the moduli space defined in Definition 1.6, the main tool comes from the theory of moment maps in symplectic geometry [Aud04]. From now on, we will concentrate on the trivial bundle $\mathbb{P}^1 \times \mathbb{C}^n$.

2.1 Symplectic Nature of the Moduli Space

We first take a glance at what have been moduled by the gauge transformation group. To be simplify, we assume a meromorphic connection has an expression near some pole:

$$\nabla = d - A = d - \left(\frac{A_k}{z^k} + \cdots + \frac{A_1}{z} + HP \right) dz$$

where HP means the holomorphic part, ∇ is formal equivalent to A^0 implies that there exists a $g(z) \in GL_n(\mathbb{C}[\![z]\!])$ such that

$$gAg^{-1} + (dg)g^{-1} = A^0$$

If we write $g(z)$ as

$$g(z) = g_0 + g_1 z + \cdots + g_{k-1} z^{k-1} + \cdots$$

where $g_0 \in GL_n(\mathbb{C})$ and $g_i \in \mathrm{End}(\mathbb{C}^n)$, its inverse can be denoted by

$$g^{-1}(z) = g_0^{-1} + \cdots$$

hence we have:

$$gAg^{-1} + (dg)g^{-1} = \left(g_0 + \cdots + g_{k-1} z^{k-1} + \cdots\right) \left(\frac{A_k}{z^k} + \cdots + \frac{A_1}{z} + HP \right) \left(g_0^{-1} + \cdots\right) dz$$

the principal part of which is

$$\left(g_0 + \cdots + g_{k-1} z^{k-1}\right) \left(\frac{A_k}{z^k} + \cdots + \frac{A_1}{z} \right) \left(g_0 + \cdots + g_{k-1} z^{k-1}\right)^{-1} dz$$

Later, we will find those $g_0 + \cdots + g_{k-1} z^{k-1}$ form a Lie group, called the **jet group** G_k, and $(A_k/z^k + \cdots + A_2/z^2)dz$ lies exactly in the dual Lie algebra $\mathfrak{g}_k^*$ of G_k.

As we can see, if a meromorphic connection is formal equivalent to A^0 near some pole, it must lie in the coadjoint orbit of the jet group G_k which containing the nice formal form.

So let's start from the jet group.

Definition 2.1 (Jet Group [Boa99])**.** The $(k-1)$-jet group is defined as

$$G_k = GL_n\left(\mathbb{C}[\zeta]/\zeta^k\right)$$

Remark 2.1. (1) An element g in G_k is an invertible matrix with entries of truncated polynomials:

$$g(\zeta) = g_0 + g_1\zeta + \cdots + g_{k-1}\zeta^{k-1}$$

where $g_0 \in GL_n(\mathbb{C})$ and no invertible conditions on the rest coefficients.

(2) An element X of the Lie algebra $\mathfrak{g}_k$ of G_k can be denoted by

$$X = X_0 + X_1\zeta + \cdots + X_{k-1}\zeta^{k-1}$$

where $X_i \in \mathrm{End}(\mathbb{C}^n)$. Hence it is an n^{2k} dimensional complex vector space, and G_k is a n^{2k} dimensional complex Lie group.

(3) An element A in the dual Lie algebra $\mathfrak{g}_k^*$ is suggested to be write as

$$A = \left(\frac{A_k}{\zeta^k} + \cdots + \frac{A_1}{\zeta}\right) d\zeta$$

where $d\zeta$ is just a symbolic notation, it has no meanings here. The pairing is given by

$$\langle A, X\rangle = \sum_{i=1}^{k} \mathrm{tr}(A_i X_{i-1}) := \mathrm{Res}_0(\mathrm{Tr}(A)\cdot X)$$

(4) There is a unipotent subgroup of G_k, namely

$$B_k = \{g(z) \in G_k : g(0) = I_n\}$$

where I_n is a identity matrix, a simple observation yields

$$G_k = GL_n(\mathbb{C}) \ltimes B_k$$

hence $GL_n(\mathbb{C})$ is also a subgroup of G_k. Moreover, it induces a decomposition on the level of Lie algebras

$$\mathfrak{g}_k = \mathfrak{gl}_n(\mathbb{C}) \oplus \mathfrak{b}_k$$

It is no wonder that we have in the dual level:

$$\mathfrak{g}_k^* = \mathfrak{gl}_n^*(\mathbb{C}) \oplus \mathfrak{b}_k^*$$

where the projection is given by

$$\mathfrak{b}_k^* \xleftarrow{\pi_{\text{irr}}} \mathfrak{g}_k^* \xrightarrow{\pi_{\text{Res}}} \mathfrak{gl}_n^*(\mathbb{C})$$

where π_{irr} and π_{Res} are defined by taking irregular part and residue part respectively.

The Lie group G_k acts on $\mathfrak{g}_k^*$ by the coadjoint action $g \cdot A := gAg^{-1}$, consequently, the coadjoint orbit O containing the element $A_k/\zeta^k + \cdots + A_2/z^2$ is a symplectic manifold. The symplectic structure is given as follows:

Lemma 2.1 ([Boa99]). (1) For any $A \in O$, the tangent space is

$$T_A O = \{[A, X] : X \in \mathfrak{g}_k\} \subset \mathfrak{g}_k^*$$

(2) The symplectic structure is given by

$$\omega_A([A, X], [A, Y]) = \langle A, [X, Y] \rangle$$

Proof . For the first statement, for any $X \in \mathfrak{g}_k$, the exponential map $\exp tX$ is an element in G_k for $t \in \mathbb{R}$, and $\gamma(t) = (\exp tX)A(\exp tX)^{-1}$ defines a curve on O passing through A whenever $t = 0$, hence an element of $T_A O$ can be expressed by

$$\left.\frac{d}{dt}\right|_{t=0} (\exp tX)A(\exp tX)^{-1} = [A, X]$$

Then the second statement follows directly form chapter 1. ■

Since $GL_n(\mathbb{C})$ is a subgroup of G_k, it also can act on O by coadjoint action, and this action is obviously Hamiltonian with residue map as the moment map.

Lemma 2.2 ([Boa99]). The Lie group action $GL_n(\mathbb{C}) \times O \longrightarrow O$ is a Hamiltonian action with the moment map

$$\mu : O \longrightarrow \mathfrak{gl}_n^*(\mathbb{C})$$
$$\left(\frac{A_k}{\zeta^k} + \cdots + \frac{A_1}{\zeta}\right) d\zeta \mapsto \frac{A_1}{\zeta} d\zeta$$

Proof. Since the moment map of the action $G_k \times O \longrightarrow O$ is just inclusion, and dual map of the inclusion $GL_n(\mathbb{C}) \hookrightarrow G_k$ is the residue map $\pi_{\text{Res}} : \mathfrak{g}_k^* \twoheadrightarrow \mathfrak{gl}_n^*(\mathbb{C})$, and since the $GL_n(\mathbb{C})$ action is induced from the inclusion, the moment map is just the composition

$$\mu : O \hookrightarrow \mathfrak{g}_k^* \xrightarrow{\pi_{\text{Res}}} \mathfrak{gl}_n^*(\mathbb{C}) \qquad ■$$

So now, the symplectic quotient $O /\!\!/ \mathrm{GL}_n(\mathbb{C})$ is a symplectic manifold.

Theorem 2.1 ([Boa99]). The moduli space $\mathcal{M}^*(\boldsymbol{A})$ of generic meromorphic connections on $\mathbb{P}^1 \times \mathbb{C}^n$ with formal form $\boldsymbol{A} = ({}^1A^0, ..., {}^mA^0)$ near each pole a_i is isomorphic to the symplectic quotient:

$$\mathcal{M}^*(\boldsymbol{A}) \cong O_1 \times ... \times O_m /\!\!/ GL_n(\mathbb{C})$$

where O_i is the coadjoint orbit of G_{k_i} containing ${}^iA^0$.

Hence there exists an intrinsic symplectic structure on $\mathcal{M}(\boldsymbol{A})$.

Proof. The proof is simple. We choose a coordinate chart on $\mathbb{P}^1$ such that the coordinate of each pole a_i is a finite complex number which will still be denoted by a_i, hence we can write down the local expression of ∇ as

$$\nabla = d - \sum_{i=1}^{m} \left(\frac{{}^iA_{k_i}}{(z-a_i)^{k_i}} + \cdots + \frac{{}^iA_1}{(z-a_i)} \right) dz$$

It contains no holomorphic part, since there are no other poles in the other coordinate chart on $\mathbb{P}^1$.

A key observation is that if ∇ is formal equivalent to $d - {}^iA^0$, it must lies in the coadjoint orbit of G_{k_i} containing ${}^iA^0 \in \mathfrak{g}^*_{k_i}$, however, the inverse is not true in general [BJL79], but it is true for the generic case [BV83].

Moreover, since ∞ is not a pole, it requires

$$^1A_1 + \cdots + {}^mA_1 = 0$$

by Lemma 2.2, this is equivalent to say $\mu({}^1A, ..., {}^mA) = 0$, and different global trivialization on left hand side implies the coadjoint action on the right hand side, hence by our discussion above, the theorem is proved. ■

2.2 Extended Moduli Space

Again, we first take a glance at what have been moduled by the gauge transformation group in the extended case. With the notation used above, $d - A$ has irregular type A^0 implies there exists a $g(z) \in \mathrm{GL}_n(\mathbb{C}[\![z]\!])$ such that the irregular part of $gAg^{-1} + (dg)g^{-1}$ is same as A^0. Hence we have:

$$gAg^{-1} + (dg)g^{-1} = \left(g_0 + \cdots + g_{k-1}z^{k-1} + \cdots\right) \left(\frac{A_k}{z^k} + \cdots + \frac{A_1}{z} + HP \right) \left(g_0^{-1} + \cdots\right) dz + ...$$

the irregular part of which is

$$\pi_{\mathrm{irr}} \left(g_0 \left(\frac{A_k}{z^k} + \cdots + \frac{A_1}{z} \right) g_0^{-1} dz \right) = b \left(\frac{A_k}{z^k} + \cdots + \frac{A_2}{z^2} \right) b^{-1} dz$$

for some $b \in B_k$, the right hand side lies in the coadjoint orbit of the subgroup B_k containing $(A_k/z^k + \cdots + A_2/z^2)dz \in \mathfrak{b}_k^*$, such an orbit will be denoted by O_B. The analysis above motivates us to define the *extended orbits*:

Definition 2.2 (Extended Coadjoint Orbit [Boa99]). The extended orbit $\tilde{O} \subset \mathrm{GL}_n(\mathbb{C}) \times \mathfrak{g}_k^*$ associated to O_B is

$$\left\{(g_0, A) \in \mathrm{GL}_n(\mathbb{C}) \times \mathfrak{g}_k^* : \pi_{\mathrm{irr}}(g_0 A g_0^{-1}) \in O_B\right\}$$

Note that we never defined a symplectic structure on $\tilde{O}$, so, next of our works will focus on the symplectic structure on $\tilde{O}$.

Lemma 2.3 ([Boa99]). Define the B_k action on $T^*G_k \times O_B$ by

$$b \cdot (g, A, B) \mapsto (bg, A, bBb^{-1})$$

where $(g, A) \in T^*G_k \cong G_k \times \mathfrak{g}_k^*$, $B \in O_B$. Then, this action is Hamiltonian and we have

$$T^*G_k \times O_B /\!\!/ B_k \cong \tilde{O}$$

Hence $\tilde{O}$ has is a symplectic manifold with the symplectic structure induced from the product space.

proof. It is easily to check that the B_k action has the moment map:

$$\mu : T^*G_k \times O_B \longrightarrow \mathfrak{b}_k^*$$

$$(g, A, B) \mapsto -\pi_{\mathrm{irr}}(\mathrm{Ad}_g^*(A)) + B$$

Now, define the map

$$\chi : \mu^{-1}(0) \longrightarrow \tilde{O} \qquad (g, A, B) \mapsto (g(0), A)$$

it is obviously well-defined surjective and with B_k as fibres. ∎

Lemma 2.4 ([Boa99]). A tangent vector $v \in T_{(g_0,A)}\tilde{O} \subset \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{g}_k^*$ of $\tilde{O}$ at (g_0, A) has the form

$$v = \left(g_0 X_0, [A, X] + g_0^{-1}\Lambda g_0\right) \in \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{g}_k^*$$

where $X = X_0 + X_1\zeta + \cdots + X_{k-1}\zeta^{k-1} \in \mathfrak{g}_k$, $\Lambda \in \mathfrak{t}^*$ (technically, it is $\Lambda d\zeta/\zeta$).

proof. For $(g, A, B) \in \mu^{-1}(0) \subset T^*G_k \times O_B = G_k \times \mathfrak{g}_k^* \times O_B$, from the proof of Lemma 2.3, we need to request

$$B = \pi_{\mathrm{irr}}\left(gAg^{-1}\right)$$

and recall that

$$T_{(g,A,B)}\mu^{-1}(0) \in T_g G_k \times \mathfrak{g}_k^* \times T_B O_B \cong \mathfrak{g}_k \times \mathfrak{g}_k^* \times T_B O_B$$

now, from lemma 2.3, the map

$$\chi : \mu^{-1}(0) \longrightarrow \tilde{O} \qquad (g, A, B) \mapsto (g_0, A)$$

is surjective, hence the tangent map

$$(d\chi)_{(g,A,B)} : T_{(g,A,B)}\mu^{-1}(0) \longrightarrow T_{(g_0,A)}\tilde{O}$$

is also surjective. Notice that, for $X = X_0 + X_1\zeta + \cdots + X_{k-1}\zeta^{k-1} \in \mathfrak{g}_k$, and $\Lambda \in \mathfrak{t}^*$, the curve defined by

$$\gamma(t) = \left(ge^{tX}, e^{-tX}\left(A + tg_0^{-1}\Lambda g_0\right)e^{tX}, B\right) : I \longrightarrow \mu^{-1}(0)$$

is a curve in $\mu^{-1}(0)$ passing through (g, A, B) whenever $t = 0$, indeed,

$$\begin{aligned}\pi_{\text{irr}}\left(ge^{tX}e^{-tX}\left(A + tg_0^{-1}\Lambda g_0\right)e^{tX}e^{-tX}g\right) &= \pi_{\text{irr}}\left(gAg^{-1} + tgg_0^{-1}\Lambda g_o g^{-1}\right)\\ &= \pi_{\text{irr}}\left(gAg^{-1}\right) = B\end{aligned}$$

so, a tangent vector $\beta \in T_{(g,A,B)}\mu^{-1}(0)$ can be expressed by

$$\beta = \left.\frac{d}{dt}\right|_{t=0}\gamma(t) = \left(gX, [A, X] + g_0^{-1}\Lambda g_0, 0\right)$$

hence any $v \in T_{(g_0,A)}\tilde{O}$ can be expressed by

$$\begin{aligned}v := (d\chi)_{(g,A,B)}\beta &= \left.\frac{d}{dt}\right|_{t=0}\chi\circ\gamma(t)\\ &= \left.\frac{d}{dt}\right|_{t=0}\left(g_0e^{tX_0}, e^{-tX}\left(A + tg_0^{-1}\Lambda g_0\right)e^{tX}\right)\\ &= \left(g_0X_0, [A, X] + g_0^{-1}\Lambda g_0\right)\end{aligned}$$

as was to be shown. ■

Remark 2.2. If we use the tangent map of left multiplication

$$\left(dL_{g^{-1}}\right)_g : T_g G_k \longrightarrow \mathfrak{g}_k$$

to identify T_gG_k with $\mathfrak{g}_k$, then our tangent vector $v \in T_{(g_0,A)}\tilde{O}$ can be identified by

$$v = \left(X_0,\, [A, X] + g_0^{-1}\Lambda g_0\right) \in \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{g}_k^*$$

similarly, a tangent vector $\beta \in T_{(g,A,B)}\mu^{-1}(0)$ then becomes to

$$\beta = \left(X,\, [A, X] + g_0^{-1}\Lambda g_0, 0\right) \in \mathfrak{g}_k \times \mathfrak{g}_k^* \times T_BO_B$$

and we will use this expression in the later computation.

Lemma 2.5 ([Boa99]). For $v_1, v_2 \in T_{(g_0,A)}\tilde{O}$, the symplectic form on $\tilde{O}$ is given by

$$\tilde{\omega}_{(g_0,A)}(v_1, v_2) = \left\langle \Lambda_1, g_0X_2g_0^{-1}\right\rangle - \left\langle \Lambda_2, g_0X_1g_0^{-1}\right\rangle + \langle A, [X_1, X_2]\rangle$$

where $v_i = \left(X_{i_0},\, [A, X_i] + g_0^{-1}\Lambda_i g_0\right)$

proof. Recall that for $\beta_1, \beta_2 \in T_{(g,A,B)}\mu^{-1}(0) \subset \mathfrak{g}_k \times \mathfrak{g}_k^* \times T_BO_B$, the symplectic form on $\mu^{-1}(0) \subset T^*G_k \times O_B$ is

$$\begin{aligned}
\omega_{(g,A,B)}(\beta_1, \beta_2) &= \omega_{(g,A)}\left(\mathrm{pr}_1^*\beta_1, \mathrm{pr}_1^*\beta_2\right) + \omega_B\left(\mathrm{pr}_2^*\beta_1, \mathrm{pr}_2^*\beta_2\right) \\
&= \omega_{(g,A)}\left(\left(X_1,\, [A, X_1] + g_0^{-1}\Lambda_1 g_0\right), \left(X_2,\, [A, X_2] + g_0^{-1}\Lambda_2 g_0\right)\right) \\
&= \left\langle [A, X_1] + g_0^{-1}\Lambda_1 g_0,\, X_2\right\rangle - \left\langle [A, X_2] + g_0^{-1}\Lambda_2 g_0,\, X_1\right\rangle - \langle A,\, [X_1, X_2]\rangle \\
&= \left\langle \Lambda_1,\, g_0X_2g_0^{-1}\right\rangle - \left\langle \Lambda_2,\, g_0X_1g_0^{-1}\right\rangle - \langle A,\, [X_1, X_2]\rangle
\end{aligned}$$

where pr_i is the projection of $T^*G_k \times O_B$ onto the first and second component with respect to $i = 1, 2$ and pr_i^* represents for their tangent map, and

$$\beta_i = \left(X_i,\, [A, X_i] + g_0^{-1}\Lambda_i g_0, 0\right), \qquad i = 1, 2$$

hence the induced symplectic form on $\tilde{O}$ is

$$\begin{aligned}
\tilde{\omega}_{(g_0,A)}(v_1, v_2) &= \tilde{\omega}_{(g_0,A)}\left((d\chi)_{(g,A,B)}v_1, (d\chi)_{(g,A,B)}v_2\right) \\
&= \omega_{(g,A,B)}(\beta_1, \beta_2) \\
&= \left\langle \Lambda_1,\, g_0X_2g_0^{-1}\right\rangle - \left\langle \Lambda_2,\, g_0X_1g_0^{-1}\right\rangle - \langle A,\, [X_1, X_2]\rangle
\end{aligned}$$

as was to be shown. ■

We can see that an element $(g_0, A) \in \tilde{O}$ is determined by g_0, the residue and the irregular part of A, hence we have the following decoupling lemma:

Lemma 2.6 (Decoupling [Boa99]). The following map is a symplectic isomorphism

$$\varphi : \tilde{O} \longrightarrow T^*\mathrm{GL}_n(\mathbb{C}) \times O_B$$
$$(g_0, A) \mapsto \left(g_0, \pi_{\mathrm{Res}}(A), \pi_{\mathrm{irr}}\left(g_0 A g_0^{-1}\right)\right)$$

proof. The map

$$\psi : T^*\mathrm{GL}_n(\mathbb{C}) \times O_B \longrightarrow \tilde{O}$$
$$(g_0, S, B) \mapsto \left(g_0,\ g_0^{-1} B g_0 + S\right)$$

is the inverse of φ, and we can show it is symplectic by straightforward computation. ■

Lemma 2.7 ([Boa99]). The Lie group action

$$\mathrm{GL}_n(\mathbb{C}) \times \tilde{O} \longrightarrow \tilde{O}$$
$$h \cdot (g_0, A) \mapsto \left(g_0 h^{-1},\ hAh^{-1}\right)$$

is a free Hamiltonian action, the moment map is

$$\mu : \tilde{O} \longrightarrow \mathfrak{gl}_n^*(\mathbb{C})$$
$$(g_0, A) \mapsto -\pi_{\mathrm{Res}}(A)$$

proof. The induced $\mathrm{GL}_n(\mathbb{C})-$action on the decoupling $T^*\mathrm{GL}_n(\mathbb{C}) \times O_B$ is

$$h \cdot (g_0, S, B) = \left(g_0 h^{-1}, hSh^{-1}, B\right)$$

now, we need to check whether $\mathrm{GL}_n(\mathbb{C})-$action on $T^*\mathrm{GL}_n(\mathbb{C})$ is Hamiltonian. For $X \in \mathfrak{gl}_n(\mathbb{C})$, if we identify $T_g\mathrm{GL}_n(\mathbb{C})$ with $\mathfrak{gl}_n(\mathbb{C})$ by the left multiplication, we can compute the fundamental vector field

$$\underline{X}(g, S) = (-X, [X, S]) \in \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n^*(\mathbb{C})$$

hence for any $(Y, R) \in T_{(g,S)}T^*\mathrm{GL}_n(\mathbb{C}) \cong \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n^*(\mathbb{C})$, we can compute

$$(\iota_{\underline{X}}\omega)_{(g,S)}(Y, R) = \langle [X, S],\ Y\rangle + \langle R, X\rangle + \langle S, [X, Y]\rangle$$
$$= \langle R,\ X\rangle$$

next we claim

$$\mu^1 : T^*\mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathfrak{gl}_n^*(\mathbb{C}) \qquad (g, S) \mapsto -A$$

is the moment map, in fact, for any $X \in \mathfrak{gl}_n(\mathbb{C})$, we can compute the tangent map of μ_X^1

straightforward by

$$\left(d\mu^1_X\right)_{(g,S)}(Y,R) = -\langle R, X\rangle$$

hence it is a Hamiltonian action, thus we can compute the moment map by

$$\mu(g_0, A) = \mu^1\left(\pi_{\mathrm{Res}}(A)\right) + 0 = -\pi_{\mathrm{Res}}\left(A\right)$$

as was to be shown ■

Now, using an analogue method from Theorem 2.1, we can prove:

Theorem 2.2 ([Boa99])**.** The extended moduli space is isomorphic to :

$$\tilde{\mathcal{M}}^*(\boldsymbol{A}) \cong \tilde{O}_1 \times \cdots \times \tilde{O}_m /\!\!/ GL_n(C)$$

3 Stokes Representations and Monodromy Manifolds

As we mentioned in section 1.2, there is an important family of invariants in $\widehat{\mathrm{Syst}}(A^0)$, namely the Stokes factors (matrices). Meanwhile, the local theory of meromorphic connections is just the theory of linear ODEs, hence Stokes factors can help us to investigate the moduli space of meromorphic connections as well.

In this chapter, we will introduce a more generalised notion of monodromy representations, which is called the **Stokes Representation** of a groupoid $\tilde{\Gamma}$, which is a slightly larger notion than the fundamental group $\pi_1(\mathbb{P}^1 \setminus \{a_1, ..., a_m\})$.

We will define the moduli space of the Stokes representations

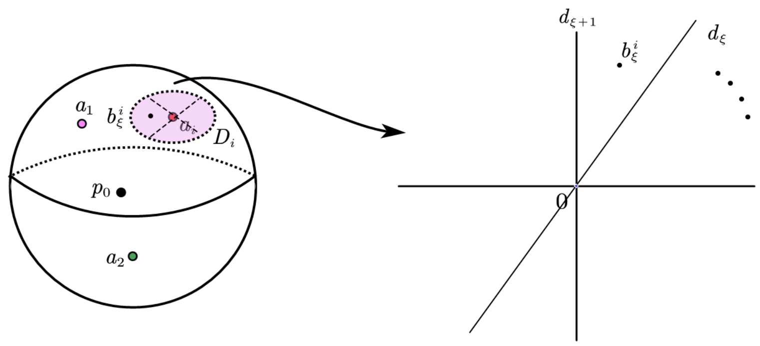
$$\tilde{M}(\boldsymbol{A}) = \mathrm{Hom}_{\mathrm{Sto}}\left(\tilde{\Gamma}; \mathrm{GL}_n(\mathbb{C})\right)/\mathrm{Gl}_n(\mathbb{C})$$

and see the relation between which and $\tilde{\mathcal{M}}(\boldsymbol{A})$ defined in the last chapter by showing two generic connections are equivalent precisely if they induce the same Stokes representations. Then we shall give an explicit description of $\tilde{M}(\boldsymbol{A})$

3.1 Stokes Representations

First, as we did before, we fix the data $\boldsymbol{A} = \{{}^1A^0, ..., {}^mA^0\}$ of irregular types near each pole a_i on $\mathbb{P}^1$, and choose m disjoint open disks $D_i \subset \mathbb{P}^1$ which containing a_i for each i so that the coordinate chart on D_i vanishes at a_i.

Now, choose a base point $p_0 \in \mathbb{P}^1 \setminus \{a_1, ..., a_m\}$, and a point b^i_ξ in each of the sectors bounded by two anti-Stokes directions at each pole a_i, where ξ ranges over some finite set which indexing the Stokes sectors of each ${}^iA^0$.



The choice of b^i_ξ

Let B_i represents for the set of those b^i_ξ near each a_i, and $\tilde{B}_i$, the lifting of B_i into the universal cover of $D_i \setminus \{a_i\}$, hence

$$\tilde{B}_i \subset \widetilde{D_i \setminus \{a_i\}} \cong \mathbb{R}$$

If $\tilde{p} \in \tilde{B}_i$, the corresponding point in B_i will be denoted by p.

Definition 3.1 (The groupoid $\tilde{\Gamma}$)**.** The groupoid $\tilde{\Gamma}$ is a category consists of following data:

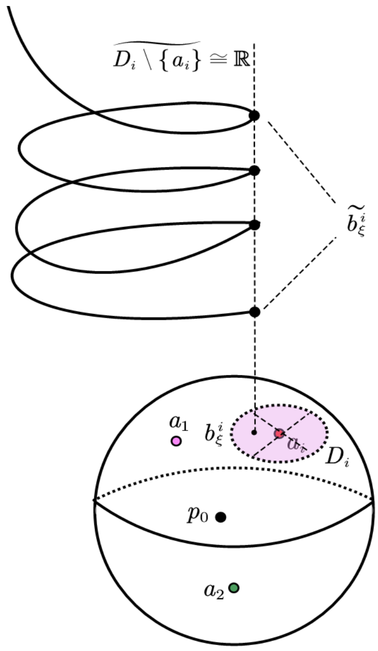
(1). The objects $\mathrm{Ob}(\tilde{\Gamma})$ is the set

$$\tilde{B} = \{p_0\} \bigcup_{i=1}^{m} \tilde{B}_i$$

(2). The morphisms between 2 objects $\tilde{p_1}, \tilde{p_2}$ are defined as the set of homotopy classes of paths

$$\gamma : [0,1] \longrightarrow \mathbb{P}^1 \setminus \{a_1, ..., a_m\}$$

from p_1 to p_2.



We assume $(E, \nabla, \boldsymbol{g})$ is a generic meromorphic connection with the compatible framing $\boldsymbol{g}$ on a holomorphic vector bundle E, with prescribed poles on the divisor $D = \sum_i k_i a_i$, and with irregular types $\boldsymbol{A}$ near each pole a_i.

We assume $\nabla = d - {}^iA$ in some local trivialization of E on D_i, hence it determines a linear ODE $dy = {}^iAy$, it has a fundamental solution, denoted by Φ_i.

$(E, \nabla, \boldsymbol{g})$ will induce a representation of the groupoid $\tilde{\Gamma}$ as follows.

Suppose $[\gamma_{\tilde{p_2}\tilde{p_1}}]$ is a morphism in $\tilde{\Gamma}$, ∇ will induce a basis of $\nabla-$horizental ($\nabla s = 0$) sections of E restricted on some neighbourhood of p_i, namely $\Phi : \mathbb{C}^n \longrightarrow E$, by extending Φ_1 analytically (as solutions of ∇) along the the path $\gamma_{\tilde{p_2}\tilde{p_1}}$ to p_2, the result will be different from Φ_2 by a constant matrix $\Phi_1 = \Phi_2 \cdot C$, hence it defines a representation:

$$\rho\left([\gamma_{\tilde{p_2}\tilde{p_1}}]\right) := C = \Phi_2^{-1} \cdot \Phi_1 \in \mathrm{GL}_n(\mathbb{C})$$

Clearly ρ only depends on the homotopy class of the path in $\mathbb{P}^1 \setminus \{a_1, ..., a_m\}$, and it is indeed a representation. It has the following properties:

Lemma 3.1 ([Boa99]). (1). For any i, if $\tilde{p_1} \in \tilde{B}_i$ and $\tilde{p_2}$ is the next element in $\tilde{B}_i$ in the positive sense, assume $\gamma_{\tilde{p_2}\tilde{p_1}}$ is a small arc in D_i from p_1 to p_2, then

$$\rho\left(\gamma_{\tilde{p_2}\tilde{p_1}}\right) \in \mathrm{Sto}_d\left({}^iA^0\right)$$

where d is the unique anti-Stokes ray that $\gamma_{\tilde{p_2}\tilde{p_1}}$ crosses.

(2). For each i, there is a diagonal matrix ${}^i\Lambda$ (which has distinct eigenvalues mod $\mathbb{Z}$ when $k_i = 1$) such that for any $\tilde{p_1} \in \tilde{B}_i$, $\tilde{p_2} \in \tilde{B}$ and $\gamma_{\tilde{p_2}\tilde{p_1}}$, we have:

$$\rho\left(\gamma_{\tilde{p_2}(\tilde{p_1}+2\pi)}\right) = \rho\left(\gamma_{\tilde{p_2}\tilde{p_1}}\right) \cdot e^{2\pi\sqrt{-1}\cdot{}^i\Lambda}$$

here $\rho\left(\gamma_{\tilde{p_2}(\tilde{p_1}+2\pi)}\right)$ is the same path as $\gamma_{\tilde{p_2}\tilde{p_1}}$, but $\tilde{p_1} + 2\pi$ is the next point after $\tilde{p_1}$ in the universal cover $\tilde{B}_i$.

proof. The 1st statement comes directly from Definition 1.9.

For (2), we assume $\nabla = d - {}^iA$ in some local trivialization on D_i, where

$$(dg)\, g^{-1} + g\,{}^iAg^{-1} = d\left({}^iQ\right) + \frac{{}^i\Lambda}{z}dz$$

where $g \in \mathrm{GL}_n(\mathbb{C}[\![z]\!])$, and $g(0) = {}^ig$ is the compatible frame at the pole a_i, $d({}^iQ)$ is the irregular part of ${}^iA^0$. Recall that two consecutive branches of $\log z$ will be differed by a $2\pi \cdot \sqrt{-1}$, hence

$$\rho\left(\gamma_{\tilde{p_2}(\tilde{p_1}+2\pi)}\right) = \rho\left(\gamma_{\tilde{p_2}\tilde{p_1}}\right) \cdot e^{2\pi\sqrt{-1}\cdot{}^i\Lambda} \quad \blacksquare$$

Remark 3.1. Since the trace $\mathrm{Tr}({}^i\Lambda)$ is the residue of ∇ at the pole a_i, by Lemma 1.1, we need to impose that

$$\sum_{i=1}^{m} \mathrm{Tr}\left({}^i\Lambda\right) = -\deg E$$

Next, we shall call a representation of $\tilde{\Gamma}$ with these 2 properties the **Stokes representation**

Definition 3.2 (Stokes Representation [Boa99]). A Stokes representation ρ is a representation of the groupoid $\tilde{\Gamma}$:

$$\rho : \tilde{\Gamma} \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

together with a choice of m diagonal matrices ${}^i\Lambda$, such that (1) and (2) in Lemma 3.1 holds.

The matrices ${}^i\Lambda$ associated with the Stokes representation are called the **exponents of**

formal monodromy, the number

$$\deg(\rho) := \sum_{i=1}^{m} \operatorname{Tr}\left({}^{i}\Lambda\right)$$

is called the **degree** of the representation (by Remark 3.1, it is a integer number).

Remark 3.2. The connections are not needed in the definition of Stokes representation appeared above, in fact, once we get a choice of nice formal form $\boldsymbol{A}$ near each pole a_i together with a choice of formal monodromy $\boldsymbol{\Lambda}$, a Stokes representation ρ can be defined. Hence to be simplified, we can use $(\rho, \boldsymbol{A}, \boldsymbol{\Lambda})$ to express for a Stokes representation.

The collection of all Stokes representations of $\tilde{\Gamma}$ is denoted by $\mathrm{Hom}_{\mathrm{Sto}}(\tilde{\Gamma}; \mathrm{GL}_n(\mathbb{C}))$. Next, we will define a $\mathrm{GL}_n(\mathbb{C})$ action on $\mathrm{Hom}_{\mathrm{Sto}}(\tilde{\Gamma}; \mathrm{GL}_n(\mathbb{C}))$.

Definition 3.3 ([Boa99])**.** Suppose $\tilde{p_1}, \tilde{p_2} \in \tilde{B} \setminus \{p_0\}$, $g \in \mathrm{GL}_n(\mathbb{C})$, we define

$$(g \cdot \rho)\left(\gamma_{p_0 p_0}\right) = g\rho\left(\gamma_{p_0 p_0}\right) g^{-1} \qquad (g \cdot \rho)\left(\gamma_{p_0 \tilde{p_1}}\right) = g\rho\left(\gamma_{p_0 \tilde{p_1}}\right)$$

$$(g \cdot \rho)\left(\gamma_{\tilde{p_2} p_0}\right) = \rho\left(\gamma_{\tilde{p_2} p_0}\right) g^{-1} \qquad (g \cdot \rho)\left(\gamma_{\tilde{p_2} \tilde{p_1}}\right) = \rho\left(\gamma_{\tilde{p_2} \tilde{p_1}}\right) g^{-1}$$

If two Stokes representations are differed by a $\mathrm{GL}_n(\mathbb{C})$ action, then we call these 2 representations are equivalent, the space of equivalent classes of Stokes representations will be denoted by

$$\tilde{M}(\boldsymbol{A}) = \mathrm{Hom}_{\mathrm{Sto}}(\tilde{\Gamma}; \mathrm{GL}_n(\mathbb{C}))/\mathrm{GL}_n(\mathbb{C})$$

The main theorem will be stated as follows:

Theorem 3.1 ([Boa99])**.** Two generic connections $(E_1, \nabla_1, \boldsymbol{g}_1), (E_2, \nabla_2, \boldsymbol{g}_2)$ $(\mathrm{rank} E_1 = \mathrm{rank} E_2 = n)$ with irregular type $\boldsymbol{A}$ are equivalent, if and only if they induce the equivalent Stokes representations. In particular, we have an injection:

$$\nu : \tilde{\mathcal{M}}(\boldsymbol{A}) \longrightarrow \tilde{M}(\boldsymbol{A})$$

this map ν is called the *Riemann-Hilbert map*

proof. The only if part comes directly from the construction of $\mathrm{GL}_n(\mathbb{C})$ action. For the if part, we denote

$$\Phi^j(z_i) : \mathbb{C}^n \longrightarrow E_j$$

the canonical basis of solutions of ∇_j on each sector at a_i, hence the local isomorphism

$$\Phi^2 \circ \left(\Phi^1\right)^{-1} : E_1 \longrightarrow E_2$$

can be extended to the whole $\mathbb{P}^1$, this is a desired isomorphism between $(E_1, \nabla_1, \boldsymbol{g}_1)$ and $(E_2, \nabla_2, \boldsymbol{g}_2)$. ■

Remark 3.3. The study of the surjectivity of the Riemann-Hilbert map ν, that is for any Stokes representation $(\rho, \boldsymbol{A}, \boldsymbol{\Lambda})$, does there exist a connection $(E, \nabla, \boldsymbol{g})$ on some vector bundle E with the irregular type $\boldsymbol{A}$ such that the representation induced by which is precisely ρ? This is the (generalised version of) Riemann-Hilbert correspondence. Later, we will see this correspondence does hold in the case of degree zero vector bundle, that is if we denoted by $\tilde{\mathcal{M}}^0(\boldsymbol{A})$ the extended moduli space of meromorphic connections on degree 0 bundles, then the Riemann-Hilbert map is an isomorphism:

$$\nu_0 : \tilde{\mathcal{M}}^0(\boldsymbol{A}) \longrightarrow \tilde{M}_0(\boldsymbol{A})$$

3.2 Explicit Monodromy Manifolds

In this part, we will give an explicit description of $\tilde{M}(\boldsymbol{A})$. Before doing this, we shall introduce the notion of the *monodromy manifolds*.

Suppose $N_1, ..., N_m$ are m manifolds, we have maps $\mu_i : N_i \longrightarrow G$ to some group G. There is a $\mathrm{GL}_n(\mathbb{C})$-action on G such that each μ_i is $\mathrm{GL}_n(\mathbb{C})-$equivariant, define a map $\boldsymbol{\mu}$ as follows:

$$\boldsymbol{\mu} : N_1 \times \cdots N_m \longrightarrow G \quad (n_1, ..., n_m) \mapsto \rho_m(n_m) \cdots \rho_1(n_1)$$

$\boldsymbol{\mu}$ is clearly $\mathrm{GL}_n(\mathbb{C})$-equivariant, we will write the quotient:

$$N_1 \times \cdots \times N_m /\!\!/ \mathrm{GL}_n(\mathbb{C}) := \boldsymbol{\mu}^{-1}(1)/\mathrm{GL}_n(\mathbb{C})$$

A manifold with such a form will called the monodromy manifold, for example, the moduli space $\mathcal{M}(\boldsymbol{A})$ and $\tilde{\mathcal{M}}(\boldsymbol{A})$ defined in the previous chapter are both monodromy manifolds.

Definition 3.4 ([Boa99]). Let $U_\pm$ be the upper/lower triangulated subgroup of $\mathrm{GL}_n(\mathbb{C})$, $\mathfrak{t}$, the set of diagonal $n \times n$ matrices and k_i is the order of the pole a_i, we define the manifold

$$\tilde{\mathcal{C}}_i := \mathrm{GL}_n(\mathbb{C}) \times (U_+ \times U_-)^{k_i - 1} \times \mathfrak{t}$$

A point of $\tilde{\mathcal{C}}_i$ will be denoted by $(C_i, {}^i\boldsymbol{S}, {}^i\Lambda')$, here

$${}^i\boldsymbol{S} = \left({}^iS_1, ..., {}^iS_{2k_i-2}\right) \in (U_+ \times U_-)^{k_i - 1}$$

The map $\mu_i : \tilde{\mathcal{C}}_i \longrightarrow GL_n(\mathbb{C})$ is defined as follows:

$$\mu_i\left(C_i, {}^i\boldsymbol{S}, {}^i\Lambda'\right) := C_i^{-1}\left({}^iS_1 \cdots {}^iS_{2k_i-2} \cdot e^{2\pi\sqrt{-1}\cdot{}^i\Lambda'}\right)C_i$$

The $\mathrm{GL}_n(\mathbb{C})$ action on $\tilde{\mathcal{C}}_i$ is given by

$$g \cdot \left(C_i, {}^i\boldsymbol{S}, {}^i\Lambda'\right) := \left(C_i g^{-1}, {}^i\boldsymbol{S}, {}^i\Lambda'\right)$$

hence it clearly makes μ_i equivariant, we define the *extended monodromy manifold* to be

$$\tilde{M} := \tilde{\mathcal{C}}_1 \times \cdots \times \tilde{\mathcal{C}}_m /\!\!/ \mathrm{GL}_n(\mathbb{C})$$

Lemma 3.2 ([Boa99]). The extended monodromy manifold $\tilde{M}$ is indeed a complex manifold with same dimension as $\tilde{\mathcal{M}}^*(\boldsymbol{A})$.

Our main goal in this section is to show that extended monodromy manifold $\tilde{M}$ isomorphic to the moduli space of Stokes representations $\tilde{M}(\boldsymbol{A})$, the construction of this isomorphism will depend on a *choice of Tentacles*, we will see later that those ${}^i\boldsymbol{S}$ come from the Stokes matrices and ${}^i\Lambda'$ from the exponents of the formal monodromy.

Definition 3.5 (A Choice of Tentacles [Boa99]). A choice of tentacles $\mathcal{T}$ is a choice of:

(1). A point p_i in **some sector** at a_i between two anti-Stokes rays.

(2). **A** lift $\tilde{p}_i$ of each p_i to the level of universal cover of the punctured disk $\widetilde{D_i \setminus \{a_i\}} \cong \mathbb{R}$.

(3). A base point $p_0 \in \mathbb{P}^1 \setminus \{a_1, ..., a_m\}$.

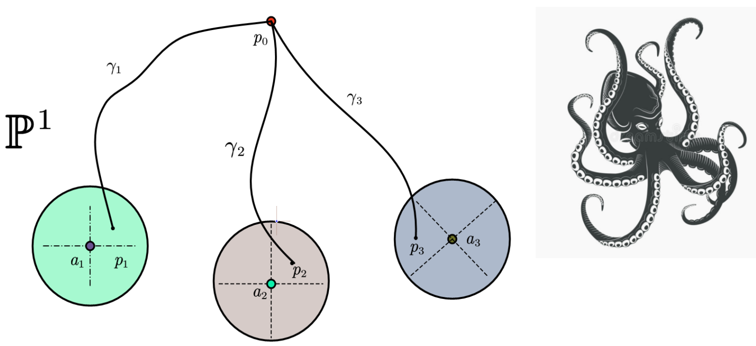
(4). For each p_i, a path

$$\gamma_i : [0,1] \longrightarrow \mathbb{P}^1 \setminus \{a_1, ..., a_m\}$$

travels from p_0 to p_i such that the loop based at p_0

$$\left(\gamma_m^{-1} \cdot \beta_m \cdot \gamma_m\right) \cdots \left(\gamma_1^{-1} \cdot \beta_1 \cdot \gamma_1\right)$$

is contractible in the fundamental group $\pi_1(\mathbb{P}^1 \setminus \{a_1, ..., a_m\}, p_0)$, where β_i is any loop in $D_i \setminus \{a_i\}$ based at p_i winding a_i once in a positive sense.



A choice of tentacles, which really looks likes a tentacle

Theorem 3.2 ([Boa99]). For each choice of tentacles $\mathcal{T}$ there is an explicit isomorphism

$$\tilde{\phi}_{\mathcal{T}}:\tilde{M}(\boldsymbol{A})\longrightarrow\tilde{M}:=\tilde{\mathcal{C}}_1\times\cdots\times\tilde{\mathcal{C}}_m/\!\!/\mathrm{GL}_n(\mathbb{C})$$

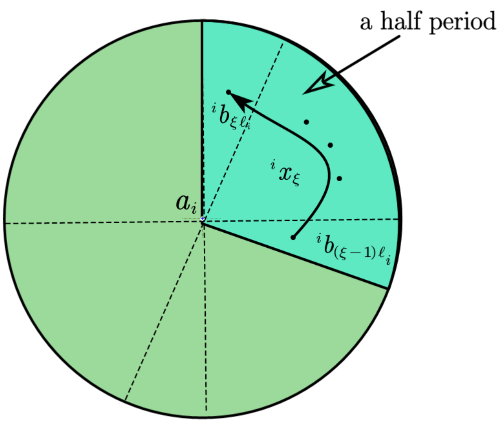
proof. For a given Stokes representation $(\rho,\boldsymbol{A},\boldsymbol{\Lambda})\in\mathrm{Hom}_{\mathrm{Sto}}(\tilde{\Gamma};\mathrm{Gl}_n(\mathbb{C}))$, we use ${}^ib_\xi\in B_i$ represents for the choice of the point in the ξ−th Stokes sector ${}^i\mathrm{Sect}_\xi$ in $D_i\setminus\{a_i\}$ during the construction of $\tilde{\Gamma}$, and without loss of generality, we assume the choice of the points p_i in the tentacles $\mathcal{T}$ were chose among one of those ${}^ib_\xi$, and p_0 coincides with the fundamental point in $\tilde{\Gamma}$.

As we have mentioned in section 1.2, the point p_i determines a labelling convention of the anti-Stokes rays of ${}^iA^0$, and its corresponding point $\tilde{p}_i$ in the universal cover determines a branch of $\log z$, hence we can determine a permutation matrix P_i (cf. Lemma 1.2) which can upper/lower-triangularise all Stokes factors in $\mathrm{Sto}_{{}^id_\xi}({}^iA^0)$.

We define:

$$C_i:=P_i^{-1}\cdot\rho\left(\gamma_{\tilde{p}_ip_0}\right)=P_i^{-1}\cdot\rho\left(\gamma_i\right)\in\mathrm{GL}_n(\mathbb{C})$$

We shall use $r_i=|{}^i\mathbb{A}|$ to represent for the number of anti-Stokes directions of ${}^iA^0$, and recall that it is divisible by $2k_i-2$, the division is denoted by $\ell_i=r_i/2(k_i-1)$.



small arc

Next, let ${}^ix_\xi$ be the morphism between the points ${}^i\tilde{b}_{(\xi-1)\ell_i}$ and ${}^i\tilde{b}_{\xi\ell_i}$, which is a small arc crossing a half-period from the sector ${}^i\mathrm{Sect}_{(\xi-1)\ell_i}$ to ${}^i\mathrm{Sect}_{\xi\ell_i}$, and define

$${}^iS_\xi:=P_i^{-1}\cdot\rho\left({}^ix_\xi\right)\cdot P_i,\quad \xi=1,...,2k_i-2$$

and

$${}^i\Lambda^{'}:=P_i^{-1}\cdot{}^i\Lambda\cdot P_i,\quad \xi=1,...,2k_i-2$$

hence a choice of tentacles $\mathcal{T}$ determines an element $\boldsymbol{C}_i:=\left(C_i,{}^i\boldsymbol{S},{}^i\Lambda'\right)$ in each $\tilde{\mathcal{C}}_i$ associated to each Stokes representation $(\rho,\boldsymbol{A},\boldsymbol{\Lambda})$, i.e a map:

$$\phi_{\mathcal{T}} : \mathrm{Hom}_{\mathrm{Sto}}\left(\tilde{\Gamma}; \mathrm{GL}_n(\mathbb{C})\right) \longrightarrow \tilde{\mathcal{C}}_1 \times \cdots \times \tilde{C}_m$$

By computing

$$\begin{aligned}
\boldsymbol{\mu}\left(\boldsymbol{C}_1, ..., \boldsymbol{C}_m\right) &= \prod_{i=1}^{m} \mu_i \left(C_i, {}^{i}\boldsymbol{S}, {}^{i}\Lambda'\right) \\
&= \prod_{i=1}^{m} \rho\left(\gamma_i^{-1} \cdot {}^{i}x_1 \cdots {}^{i}x_{2k_i-2} \cdot \gamma_i\right) \\
&:= \prod_{i=1}^{m} \rho\left(\gamma_i^{-1} \cdot \beta_i \cdot \gamma_i\right) \\
&= \rho\left(\left(\gamma_m^{-1} \cdot \beta_m \cdot \gamma_m\right) \cdots \left(\gamma_1^{-1} \cdot \beta_1 \cdot \gamma_1\right)\right) \\
&= I \in \mathrm{GL}_n(\mathbb{C})
\end{aligned}$$

hence the image of $\phi_{\mathcal{T}}$ is exactly $\boldsymbol{\mu}^{-1}(1)$, by Theorem 3.1, it also injective, and since these μ_i are clearly $\mathrm{GL}_n(\mathbb{C})-$equivariant, hence it can be descended to the quotient space:

$$\tilde{\phi}_{\mathcal{T}} : \tilde{M}(\boldsymbol{A}) \longrightarrow \tilde{M} := \tilde{\mathcal{C}}_1 \times \cdots \times \tilde{\mathcal{C}}_m /\!\!/ \mathrm{GL}_n(\mathbb{C})$$

■

In the rest part of this thesis, we will pay more attention on the case of trivial bundles, hence we only need to consider the component of degree 0 part of the moduli space, that is $\tilde{M}_0(\boldsymbol{A})$, we will give a symplectic structure on $\tilde{M}_0(\boldsymbol{A})$ but from a different approach, namely, the C^{∞} approach.

4 The C^∞ Approach

4.1 The Flat Singular C^∞ Connections

Let $D = k_1 a_1 + \cdots k_m a_m \in \text{Div}(\mathbb{P}^1)$, $k_i > 0$, $\mathcal{C}^\infty$ be the sheaf of smooth functions on $\mathbb{P}^1$, and $\mathcal{O}_D$ the sheaf of meromorphic functions with poles on D, and we define the *sheaf of smooth functions on* $\mathbb{P}^1$ *with poles on* D to be

$$\mathcal{C}_D^\infty := \mathcal{O}_D \otimes_{\mathcal{O}} \mathcal{C}^\infty$$

and analogously, we can define the sheaf of smooth $r-$forms with poles on D, namely Ω_D^r.

Remark 4.1. Locally, if we choose a coordinate chart (D_i, z_i) containing a_i with $z_i(a_i) = 0$, a function $f \in \mathcal{C}_D^\infty(D_i)$ can be expressed by

$$f(z_i) = \frac{g}{z_i^{k_i}}$$

where g is a smooth function on D_i.

Definition 4.1 (Laurent Map [Mal66])**.** We fix a family of coordinate charts D_i near each a_i, such that the coordinate of a_i is 0, the **Laurent Map** at each D_i is defined by

$$L_i : \Omega_D^r\left(\mathbb{P}^1\right) \longrightarrow z_i^{-k_i}\mathbb{C}[\![z_i, \bar{z}_i]\!] \otimes \bigwedge^r \mathbb{C}^2$$

the Laurent expansion of the local expression of $\omega \in \Omega_D^r$ on D_i at the point a_i.

For example, $f \in \mathcal{C}_D^\infty$ has local expression $f = g/z_i^{k_i}$ on D_i, the Laurent map on f is:

$$L_i(f) = \frac{L_i(g)}{z_i^{k_i}}$$

where $L_i(g)$ is the Taylor expansion of g at $z = 0$.

Remark 4.2. It is not hard to see the following facts of the Laurent map:

(1). If $L_i(\omega) = 0$ then ω is non-singular at a_i.

(2). L_i commutes with the exterior derivative and wedge product:

$$L_i(\omega_1 \wedge \omega_2) = L_i(\omega_1) \wedge L_i(\omega_2)$$

$$dL_i = L_i d$$

Another important fact is that

Lemma 4.1 (E.Borel)**.** The Laurent map L_i is surjective. More specifically, if M is a differential manifold, $I = [-c, c] \subset \mathbb{R}$, and $\hat{f} \in \mathbb{C}[\![x, y]\!] \otimes C^\infty(M)$, where (x, y) is the coordinate on $\mathbb{C}$, then there exists an $f \in C^\infty(M \times I^2)$ such that the Taylor expansion of f at $x = y = 0$ is $\hat{f}$.

Definition 4.2 (C^∞ singular connections [Boa99])**.** A C^∞ singular connection on a C^∞ vector bundle E over $\mathbb{P}^1$ is a morphism between sheave:

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega_D^1$$

where $\mathcal{E}$ is the sheaf of C^∞ sections of E, satisfying the Leibniz rule:

$$\nabla(f \cdot s) = (df) \otimes s + f \cdot \nabla s$$

where f is a smooth function.

Remark 4.3. Locally, if we choose a local trivialization of E near the singularity a_i, ∇ has the expression:

$$\nabla = d - \frac{{}^i A}{z_i^{k_i}}$$

where ${}^i A$ is a matrix-valued C^∞ 1-form.

In this chapter, we will mainly focus on the case that E is a trivial bundle, recall that any degree 0 vector bundle E over $\mathbb{P}^1$ is $C^\infty-$trivial. Since trivial bundle admits a global C^∞ trivialization, the connection matrix α of $\nabla = d - \alpha$ is in fact a singular 1-form defined on the whole $\mathbb{P}^1$, hence the collection of all singular connection with poles on D can be denoted by

$$\mathcal{A}_D := \left\{ d - \alpha : a \in \operatorname{End}_n \left(\Omega_D^1 \left(\mathbb{P}^1 \right) \right) \right\}$$

Moreover, the gauge transformation group of a trivial bundle is simply:

$$\mathcal{G} = \operatorname{Aut}\left(\mathbb{P}^1 \times \mathbb{C}^n \right) \cong \operatorname{GL}_n \left(C^\infty \left(\mathbb{P}^1 \right) \right)$$

for $g \in \mathcal{G}$, the g action on a connection $d - \alpha \in \mathcal{A}_D$ is given by

$$g[\alpha] = (dg)g^{-1} + g\alpha g^{-1}$$

Remark 4.4. It is important to check that the gauge transformation is associative, i.e. for $h, g \in \mathcal{G}$, one has

$$h[g[\alpha]] = hg[\alpha]$$

hence it does define a Lie group action.

Definition 4.3 (Curvature [Huy05]). The *curvature* of a connection ∇ is $\Omega(\nabla) := \nabla^2$:

$$\Omega : \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega_D^1 \xrightarrow{\nabla} \mathcal{E} \otimes \Omega_D^1 \otimes \Omega_D^1 = \mathcal{E} \otimes \Omega_{2D}^2$$

a connection is called **flat** if its curvature is zero.

Hence if we write $\nabla = d - \alpha$, the curvature is

$$\Omega = (d - \alpha) \wedge (d - \alpha) = -d\alpha + \alpha \wedge \alpha$$

which is a singular 2-form, called the *curvature form*. It is no wonder that

Corollary 4.1. Every meromorphic connection is flat.

Another important fact is

Lemma 4.2. The gauge transformation of a flat connection is again flat.

proof. Suppose $\Omega(\nabla) = 0$, for $g \in \mathcal{G}$, notice that

$$\Omega(g[\nabla]) = g\Omega(\nabla)g^{-1} = 0 \qquad \blacksquare$$

Next, as we did in the case of meromorphic connections, we choose a family of nice normal (formal) form $\boldsymbol{A} = ({}^1A^0, ..., {}^mA^0)$ near each singularity a_i, where ${}^iA^0$ are generic diagonal matrices with only principal parts in Laurent expansion. In order to define a connection with formal type $\boldsymbol{A}$, we can do comparison between their Laurent expansions:

Definition 4.4 ([Boa99]). The set of all singular connections with formal type $\boldsymbol{A}$ is

$$\mathcal{A}_D(\boldsymbol{A}) := \left\{ d - \alpha : L_i(\alpha) = {}^iA^0 \right\}$$

similarly, the set of singular connections with irregular type $\boldsymbol{A}$ is

$$\tilde{\mathcal{A}}_D(\boldsymbol{A}) := \left\{ d - \alpha : L_i(\alpha) = {}^iA^0 + \left({}^i\Lambda - {}^i\Lambda^0\right)\frac{dz_i}{z_i}, \text{ for some } {}^i\Lambda \in \mathfrak{t} \right\}$$

If we impose an extra condition, flat connections, on each of the spaces in the above definition, then the corresponding space will be denoted by $\mathcal{A}_{\text{fl}}(\boldsymbol{A})$ and $\tilde{\mathcal{A}}_{\text{fl}}(\boldsymbol{A})$ respectively.

Definition 4.5. $\mathcal{G}_T, \mathcal{G}_1$ are the subgroups of $\mathcal{G}$, consisting of all elements which have Taylor expansion equal to constant diagonal matrices and identity I respectively.

4.2 The C^∞ Linear Ordinary Differential Equations

Now, we will give a C^∞ description of the space $\mathcal{H}\left(A^0\right)$ defined in section 1.2.

Let $\mathbb{D}$ be a unit disk in $\mathbb{C}$ containing $z=0$, and fix a diagonal meromorphic connection germ $d - A^0$ with an order k pole at $z=0$. Recall that in this case, our $\mathcal{G}$ is actually $\mathrm{GL}_n\left(C^\infty(\mathbb{D})\right)$, $\mathcal{G}_1$ and $\mathcal{G}_T$ are of same meaning as before.

Lemma 4.3 ([Boa99]). The projection $(A, \hat{F}) \mapsto \hat{F}$ defines an injection between $\widehat{\mathrm{Syst}}\left(A^0\right) \hookrightarrow G[\![z]\!]$.

So it allows us to identify $\widehat{\mathrm{Syst}}\left(A^0\right)$ with a subset of $G[\![z]\!]$.

Definition 4.6 ([Boa99]). We define:

$$\mathcal{F}\left(A^0\right) := L_0^{-1}\left(\widehat{\mathrm{Syst}}\left(A^0\right)\right)$$

Where $L_0 : \mathcal{G} \longrightarrow \mathrm{GL}_n(\mathbb{C}[z, \bar{z}])$ is the map of Laurent expansion at $z=0$.

Remark 4.5. For $g \in \mathcal{F}\left(A^0\right) \subset \mathcal{G}$ implies:

(1). $L_0(g) \in \mathrm{GL}_n(\mathbb{C}[\![z]\!])$, i.e no $\bar{z}-$parts in its Taylor expansion, and

(2). $L_0(g)\left[A^0\right]$ is convergent to some matrix-valued meromorphic 1-form A.

Lemma 4.4 ([Boa99]). The Taylor expansion at 0 induces isomorphisms

$$\left(\mathcal{F}\left(A^0\right)/\mathcal{G}_1\right)/\, G\{z\} \cong \mathcal{H}\left(A^0\right)$$

proof. Notice that the map

$$\begin{aligned}\mathcal{F}\left(A^0\right)/\mathcal{G}_1 &\longrightarrow \widehat{\mathrm{Syst}}\left(A^0\right)\\ [g] &\mapsto \left(L_0(g)\left[A^0\right], L_0(g)\right)\end{aligned}$$

defines a bijection (by Remark 4.2). ■

Theorem 4.1 ([BJL79]). We have isomorphism

$$\mathcal{F}\left(A^0\right)/\, G\{z\} \cong \mathcal{A}_{\mathrm{fl}}\left(A^0\right)$$

Hence combining with Lemma 4.4, we have isomorphism

$$\mathcal{H}\left(A^0\right) \cong \mathcal{A}_{\mathrm{fl}}\left(A^0\right)/\,\mathcal{G}_1$$

proof. Define a map σ by:

$$\begin{aligned}\sigma : \mathcal{F}\left(A^0\right)/\, G\{z\} &\longrightarrow \mathcal{A}_{\mathrm{fl}}\left(A^0\right)\\ [g] &\mapsto g^{-1}\left[L_0(g)\left[A^0\right]\right]\end{aligned}$$

First,we claim this map is well-defined. Indeed, the Laurent expansion of $\sigma(g)$ is

$$L_0(\sigma(g)) = L_0(g)^{-1} L_0(g) \left[A^0\right] = A^0$$

by Lemma 4.2, $\sigma(g)$ is also flat, hence $\sigma(g) \in \mathcal{A}_{\mathrm{fl}}\left(A^0\right)$.

Next, for $[g] = [g']$, which implies there exists an $h \in G\{z\}$ such that $g' = hg$, notice that $L_0(h)[A] = h[A]$, hence

$$\sigma(hg) = g^{-1} h^{-1} \left[h L_0(g) \left[A^0\right]\right] = \sigma(g)$$

so it is well-defined.

σ is also surjective. To see this, for $A \in \mathcal{A}_{\mathrm{fl}}\left(A^0\right)$, its $(0,1)-$part is non-singular, hence there exists an $g \in \mathcal{G}$ such that $(\bar{\partial} g) g^{-1} = A^{(0,1)}$. Observe that $g^{-1}[A]$ is also flat and has no $(0,1)-$part, in fact, the $(0,1)-$part of $g^{-1}[A]$ is

$$\begin{aligned}\left(\bar{\partial} g^{-1}\right) g + g^{-1} A^{(0,1)} g &= -g^{-1}\left(\bar{\partial} g\right) g^{-1} g + g^{-1}(\bar{\partial} g) g^{-1} g \\ &= -g^{-1} \bar{\partial} g + g^{-1} \bar{\partial} g = 0\end{aligned}$$

hence we can write $g^{-1}[A] = F dz/z^k$. Next we will show $L_0(g) g^{-1}[A] = A^0$, first observe that $L_0(g)$ has no $\bar{z}-$terms, in fact

$$\bar{\partial} L_0(g) = L_0(\bar{\partial} g) = L_0\left(A^{(0,1)} g\right) = 0$$

hence

$$\begin{aligned}L_0\left(g^{-1}\right)\left[A^0\right] &= L_0\left(g^{-1}\right)\left[L_0(A)\right] \\ &= L_0\left(g^{-1}[A]\right) = g^{-1}[A]\end{aligned}$$

Moreover, it also implies $g \in \mathcal{F}\left(A^0\right)$. Finally, it follows from Lemma 4.3 that σ is injective. ■

The last thing is to define the Stokes matrices from $\tilde{\mathcal{A}}_{\mathrm{fl}}(\boldsymbol{A})$ side.

For $A \in \tilde{\mathcal{A}}_{\mathrm{fl}}(\boldsymbol{A})$, from theorem 4.1, there exists an $g \in \mathcal{F}\left(A^0\right) \subset \mathcal{G}$ (with $(\bar{\partial} g) g^{-1} = A^{(0,1)}$) such that

$$g L_0\left(g^{-1}\right)\left[A^0\right] = A$$

we assume

$$A^0 = dQ + \frac{\Lambda^0}{z} dz$$

hence on each Stokes sector $\mathrm{Sect}_{\mathrm{i}}$, the fundamental solution of $dy = Ay$ can be formulated by

$$\Phi_i = g \Sigma_i\left(L_0\left(g^{-1}\right)\right) z^{\Lambda^0} e^Q$$

hence then, the notion of Stokes factors can be defined as usual as Definition 1.9.

4.3 Globalization

As the local picture was shown in section 4.2, in this section we will turn to the case on the whole $\mathbb{P}^1$.

Recall that $\boldsymbol{A} = ({}^1A^0, ..., {}^mA^0)$ are the fixed data of irregular type near each pole a_i, and we use $\tilde{\mathcal{M}}^0(\boldsymbol{A})$ represents for the extended moduli space of meromorphic connections $(E, \nabla, \boldsymbol{g})$ with irregular type $\boldsymbol{A}$ on the degree 0 bundles.

Theorem 4.2 ([Boa99]). We have the isomorphism

$$\tilde{\mathcal{M}}^0(\boldsymbol{A}) \cong \tilde{\mathcal{A}}_{\text{fl}}(\boldsymbol{A}) \Big/ \mathcal{G}_1$$

proof. The isomorphism

$$\tilde{\sigma} : \tilde{\mathcal{M}}^0(\boldsymbol{A}) \xrightarrow{\cong} \tilde{\mathcal{A}}_{\text{fl}}(\boldsymbol{A}) \Big/ \mathcal{G}_1$$

can be constructed in the following way. For $[\nabla] \in \tilde{\mathcal{M}}^0(\boldsymbol{A})$, since $\deg E = 0$, and as was shown in the local picture, we can choose a $C^\infty-$global trivialization of E, namely $g_\nabla : E \longrightarrow \mathbb{P}^1 \times \mathbb{C}^n$, such that:

(1). $g_\nabla(a_i) = {}^ig_0$ for each a_i.

(2). If we assume $\nabla = d - \alpha_g^\nabla$ under this global trivialization, the Laurent expansion at a_i is

$$L_i\left(\alpha_g^\nabla\right) = {}^iA^0$$

hence we obtain a map

$$\begin{aligned} \sigma : \tilde{\mathcal{M}}^0(\boldsymbol{A}, \boldsymbol{g}) &\longrightarrow \tilde{\mathcal{A}}_{\text{fl}}(\boldsymbol{A}) \\ [\nabla] &\mapsto d - \alpha_g^\nabla \end{aligned}$$

it is straightforward to check this σ is well-defined, surjective, and precisely has $\mathcal{G}_1-$orbit as fibre, hence we can descend it to the quotient level, i.e $\tilde{\sigma}$, which is a desired isomorphism. ■

Remark 4.6. If we restrict this $\tilde{\sigma}$ to the submanifold of trivial bundle, i.e $\tilde{\mathcal{M}}^*(\boldsymbol{A})$, then that global trivialization g_∇ can be viewed as a bundle automorphism, i.e $g_\nabla \in \mathrm{GL}_n\left(C^\infty\left(\mathbb{P}^1\right)\right)$.

As what we did in chapter 3, for any $d - \alpha \in \tilde{\mathcal{A}}_{\text{fl}}(\boldsymbol{A})$, it determines a Stokes representation:

$$\nu : \tilde{\mathcal{A}}_{\text{fl}}(\boldsymbol{A}) \longrightarrow \tilde{M}_0(\boldsymbol{A})$$

we call this ν, the monodromy map.

Theorem 4.3 (Riemann-Hilbert Correspondence [Boa99]). The monodromy map ν is surjective and precisely has the $\mathcal{G}_1-$orbit as fibre.

Hence it induces the bijection

$$\tilde{\mathcal{M}}^0(\boldsymbol{A}) \cong \tilde{\mathcal{A}}_{\mathrm{fl}}(\boldsymbol{A})/\mathcal{G}_1 \cong \tilde{M}_0(\boldsymbol{A})$$

proof. We use $\tilde{\Gamma}$ to express for the groupoid determined by $\boldsymbol{A}$, choose a tentacle $\mathcal{T}$, and thickening each γ_i by

$$\bar{\gamma}_i : [0,1] \times [0,1] \longrightarrow \mathbb{P}^1 \setminus \{a_1, ..., a_m\}$$

we denote $|\bar{\gamma}_i|$ the ribbon formed by those $\bar{\gamma}_i$. Let D_0 be the disk containing p_0 disjoint with each disk D_i containing a_i, define the region:

$$|\mathcal{T}| := \bar{D}_0 \cup \bigcup_{i=1}^{m} \left(\bar{D}_i \cup |\bar{\gamma}_i|\right) \subset \mathbb{P}^1$$

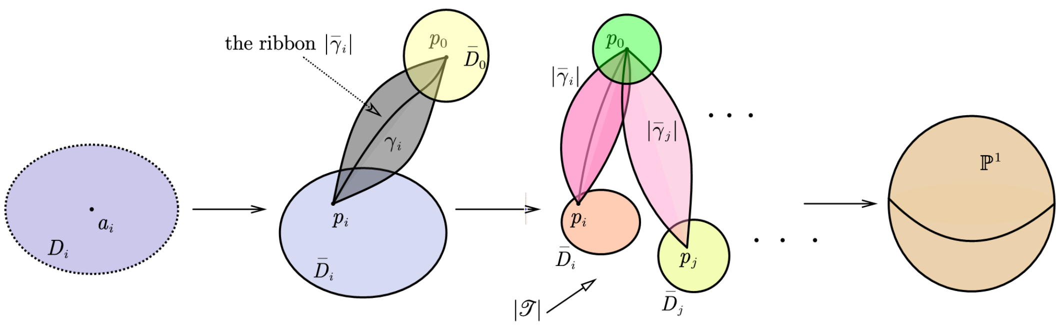
Without loss of generality, we may assume that:

(1). For each $i \neq j$, the intersection of two ribbons

$$|\bar{\gamma}_i| \cap |\bar{\gamma}_j| \subset \bar{D}_0$$

(2). $|\mathcal{T}|$ homeomorphic to a closed disk.

First, we will prove the monodromy map ν is surjective. From Theorem 3.2, every degree 0 Stokes representation determines a group of data $(\boldsymbol{C}, \boldsymbol{S}, \boldsymbol{\Lambda'})$ in the monodromy manifold, by theorem 4.1 and lemma 1.5, near each a_i there exists an $\alpha_i \in {}^i\tilde{\mathcal{A}}_{\mathrm{fl}}\left({}^iA^0\right)$ with the data of $\tilde{\mathcal{C}}_i-$component of $(\boldsymbol{C}, \boldsymbol{S}, \boldsymbol{\Lambda'})$, it is straightforward to extend α_i arbitrarily to $\bar{D}_i$. these α_i can be patched along the ribbons in the following way. Let ${}^i\Phi_0$ be the canonical



the patching of α_i

fundamental solution of α_i on the sector containing p_i, namely ${}^i\mathrm{Sect}_0$. Since $\mathrm{GL}_n(\mathbb{C})$ is path connected, we can choose a smooth map

$$\chi_i : |\bar{\gamma}_i| \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

such that $\chi_i = 1$ on $|\bar{\gamma}_i| \cap \bar{D}_0$ and $\chi_i = {}^i\Phi_0 P_i C_i$ on $|\bar{\gamma}_i| \cap {}^i\mathrm{Sect}_0$, hence we can define α on $|\mathcal{T}|$ by

$$\alpha|_{\mathcal{T}} = \begin{cases} \alpha_i & \text{on } \bar{D}_i \\ 0 & \text{on } \bar{D}_0 \\ (d\chi_i)\,\chi_i^{-1} & \text{on } |\bar{\gamma}_i| \end{cases}$$

It is straightforward to check this definition agrees on the overlaps. Now, we must extend α to the whole $\mathbb{P}^1$. The condition $\rho_m \cdots \rho_1 = 1$ guarantees that α has trivial monodromy around the boundary circle $\partial|\mathcal{T}|$, hence the local fundamental solution Ψ of α extends to a loop in $\mathrm{GL}_n(\mathbb{C})$

$$\Psi : \partial|\mathcal{T}| \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

Then $\deg\rho = 0$ implies this loop Ψ is contractible in $\mathrm{Gl}_n(\mathbb{C})$. In fact, recall that the determinant map $\det : \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathbb{C}^*$ makes $\mathrm{GL}_n(\mathbb{C})$ an $\mathrm{SL}_n(\mathbb{C})-$bundle over $\mathbb{C}^* \cong \mathbb{S}^1$, and $\mathrm{SL}_n(\mathbb{C})$ is simply connected, from the homotopy long exact sequence for fibrations, the determinant induces an isomorphism between fundamental groups

$$\pi_1\left(\mathrm{GL_n}(\mathbb{C})\right) \cong \pi_1\left(\mathbb{S}^1\right) \cong \mathbb{Z}$$

hence we need to show the induced loop

$$\det\Psi : \partial|\mathcal{T}| \longrightarrow \mathbb{C}^* \cong \mathbb{S}^1$$

has winding number zero around $z = 0$. The winding number of $\det\Psi$ is

$$\frac{1}{2\pi\sqrt{-1}} \int_{\partial|\mathcal{T}|} \frac{d\det\Psi}{\det\Psi} = \frac{1}{2\pi\sqrt{-1}} \int_{\partial|\mathcal{T}|} \mathrm{Tr}(\alpha) = \deg\rho = 0$$

thus the loop Ψ can be extended to a smooth map from the complement of $|\mathcal{T}|$ to $\mathrm{GL}_n(\mathbb{C})$. We can define $\alpha = (d\Psi)\Psi^{-1}$ on this complement and thereby obtain $\alpha \in \tilde{\mathcal{A}}_{\mathrm{fl}}(\boldsymbol{A})$, the surjectiveness hence now be proven.

As for the monodromy map ν has precisely $\mathcal{G}_1$ as fibre, it is an analogue proof of Theorem 3.1. ■

4.4 Symplectic Structure

In this section, we will give a symplectic structure on $\tilde{\mathcal{A}}_D(\boldsymbol{A})$, it is an analogue of Atiyah-Bott's frame work [AB83] on the case of non-singular connections, then by showing the curvature map is the moment map of $\mathcal{G}_1-$gauge action, hence the moduli space of *flat*

C^∞ singular connection is just the symplectic quotient

$$\tilde{\mathcal{A}}(\boldsymbol{A}) \Big/ \mathcal{G}_1 = \tilde{\mathcal{A}}_D(\boldsymbol{A}) /\!\!/ \mathcal{G}_1$$

hence a symplectic manifold. Then by taking monodromy map, $\tilde{M}_0(\boldsymbol{A})$ will inherit a symplectic structure as well.

4.4.1 The Atiyah-Bott Symplectic Form on $\tilde{\mathcal{A}}_D(\mathbf{A})$

Lemma 4.5 ([Boa99]). The space $\tilde{\mathcal{A}}_D(\boldsymbol{A})$ is a *Fréchet manifold*, and for every $\alpha \in \tilde{\mathcal{A}}_D(\boldsymbol{A})$, the tangent space at α is:

$$T_\alpha \tilde{\mathcal{A}}_D(\boldsymbol{A}) = \left\{ \phi \in \Omega^1_D\left(\mathbb{P}^1\right) \otimes \operatorname{End}(E) \middle| \, L_i(\phi) \in \mathfrak{t}\frac{dz_i}{z_i} \right\} := W$$

proof. For the last assertion only. Choose $\phi \in W$, observe that the path

$$\gamma(t) = \alpha + t\phi : I \longrightarrow \tilde{\mathcal{A}}_D(\boldsymbol{A})$$

indeed defines a path in $\tilde{\mathcal{A}}_D(\boldsymbol{A})$ with $\gamma(0) = \alpha$, hence differentiate at $t = 0$, the tangent space is W. ■

Lemma 4.6 ([Boa99]). For $\phi, \psi \in W = T_\alpha \tilde{\mathcal{A}}_D(\boldsymbol{A})$, the formula

$$\omega_\alpha(\phi, \psi) := \frac{1}{2\pi \cdot \sqrt{-1}} \int_{\mathbb{P}^1} \operatorname{Tr}(\phi \wedge \psi)$$

defines a symplectic structure on the Fréchet manifold $\tilde{\mathcal{A}}_D(\boldsymbol{A})$.

proof. Since $L_i(\phi \wedge \psi)$ is a $(2,0)-$form on $\mathbb{P}^1$ hence it is zero, by Remark 4.2 (1), $\phi \wedge \psi$ is non-singular, thus this integral is well-defined. Notice that the integration is independent of the choice of α, hence ω_α is constant, in particular, it is continuous and closed: $d\omega = 0$.

To see it is non-degenerate, if $\omega_\alpha(\phi, \psi) = 0$ for all ψ, we assume $\phi \neq 0$, it must not vanish at some $p \neq a_1, .., a_m$, hence we can construct a ψ vanishing outside a neighborhood of p such that $\omega_\alpha(\phi, \psi) \neq 0$, a desired contradiction. ■

4.4.2 The Fréchet Lie Group $\mathcal{G}$ Action is Hamiltonian

The gauge group

$$\mathcal{G} = \operatorname{GL}_n\left(C^\infty\left(\mathbb{P}^1\right)\right) = C^\infty\left(\mathbb{P}^1; \operatorname{GL}_n(\mathbb{C})\right)$$

is a Fréchet Lie group, i.e. it is group with a structure of a smooth Fréchet manifold such that the group operation and inverse are C^∞. Its Lie algebra, denoted by $\mathrm{Lie}(\mathcal{G})$, is

$$\mathrm{Lie}(\mathcal{G}) = \mathrm{End}_n\left(C^\infty\left(\mathbb{P}^1\right)\right) = C^\infty\left(\mathbb{P}^1; \mathfrak{gl}_n(\mathbb{C})\right) = \Gamma(\mathrm{End}\, E)$$

where $\Gamma(\mathrm{End}\, E)$ means the global section of the endomorphism bundle $\mathrm{End}\, E$.

For $X \in \mathrm{Lie}(\mathcal{G})$, the exponential map is then the usual exponential of matrices:

$$\begin{aligned} \exp : \mathrm{Lie}(\mathcal{G}) &\longrightarrow \mathcal{G} \\ X &\mapsto \exp X = e^X \end{aligned}$$

the exponential is also a local chart of a neighborhood of the identity, and it can be extended to a complex structure such that $\mathcal{G}$ is a complex Lie group.

Later, we will concentrate more on the subgroup $\mathcal{G}_1$ and $\mathcal{G}_T$, and it is no wonder that

$$\mathrm{Lie}(\mathcal{G}_1) = \{X \in \mathrm{Lie}(\mathcal{G}) : L_i(X) = 0\}$$

$$\mathrm{Lie}(\mathcal{G}_T) = \{X \in \mathrm{Lie}(\mathcal{G}) : L_i(X) \in \mathfrak{t}\}$$

Remark 4.7. (1). For a connection $\nabla = d - \alpha \in \tilde{\mathcal{A}}_D(\boldsymbol{A})$, the induced connection $\tilde{\nabla}$ is as follows:

$$\left(\tilde{\nabla} X\right) u = [\nabla, X] := \nabla(Xu) - X\nabla u$$

where X is a section of the endomorphism bundle $\mathrm{End}\, E$, u is the section of E. To agree with the notation in [Boa99], we denoted by d_α the induced connection of ∇, i.e:

$$d_\alpha : \Omega^0\left(\mathbb{P}^1; \mathrm{End}\, E\right) \longrightarrow \Omega^1_D\left(\mathbb{P}^1; \mathrm{End}\, E\right)$$

(2). d_α naturally induces a connection on the bundle $\mathrm{End}(E) \otimes \Omega^1_D$, which will also be denoted by d_α

$$\begin{aligned} d_\alpha : \Omega^1_D\left(\mathbb{P}^1; \mathrm{End}\, E\right) &\longrightarrow \Omega^2_{2D}\left(\mathbb{P}^1; \mathrm{End}\, E\right) \\ \phi &\mapsto d\phi + [\phi, \alpha] \end{aligned}$$

Note that the image of this d_α is actually non-singular, since we can take the Laurent expansion of $[\phi, \alpha]$, which is $(2,0)-$form on $\mathbb{P}^1$ hence zero.

Now, we can compute the fundamental vector field of $X \in \mathrm{Lie}(\mathcal{G})$:

Lemma 4.7 ([Boa99]). The Lie group $\mathcal{G}$ acts holomorphically on $\tilde{A}_D(\boldsymbol{A})$, and for $\alpha \in \tilde{\mathcal{A}}_D(\boldsymbol{A})$, and $X \in \mathrm{Lie}(\mathcal{G})$, the fundamental vector field of X associated to the gauge transformation is

$$\underline{X}(\alpha) = -d_\alpha X$$

proof. By definition we have:

$$\begin{aligned}\underline{X}(\alpha) &= \frac{d}{dt}\Big|_{t=0} \left\{(d\exp tX)(\exp tX)^{-1} + (\exp tX)\alpha(\exp tX)^{-1}\right\} \\ &= \frac{d}{dt}\Big|_{t=0} \left\{\left(de^{tX}\right)e^{-tX} + e^{tX}\alpha e^{-tX}\right\} \\ &= \frac{d}{dt}\Big|_{t=0} \left\{e^{tX}t(dX)e^{-tX} + e^{tX}\alpha e^{-tX}\right\} \\ &= dX + [X,\alpha] = -d_\alpha X \qquad \blacksquare\end{aligned}$$

Lemma 4.8 ([Boa99]). The $\mathcal{G}$ action on $\tilde{\mathcal{A}}_D(\boldsymbol{A})$ preserves the symplectic structure.

proof. Any $g \in \mathcal{G}$ will determine a differmorphism of $\tilde{\mathcal{A}}_D(\boldsymbol{A})$, we can compute its tangent map:

$$\begin{aligned}(dg)_\alpha : T_\alpha\tilde{\mathcal{A}}_D(\boldsymbol{A}) &\longrightarrow T_{g[\alpha]}\tilde{\mathcal{A}}_D(\boldsymbol{A}) \\ \phi &\mapsto g\phi g^{-1}\end{aligned}$$

In fact

$$(dg)_\alpha\phi = \frac{d}{dt}\Big|_{t=0} g[\alpha + t\phi] = g\phi g^{-1}$$

hence

$$\begin{aligned}\omega_\alpha\left((dg)_\alpha\phi, (dg)_\alpha\psi\right) &= \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{P}^1} \operatorname{Tr}\left(g\phi g^{-1} \wedge g\psi g^{-1}\right) \\ &= \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{P}^1} \operatorname{Tr}\left(\phi \wedge \psi\right) = \omega_\alpha(\phi,\psi) \qquad \blacksquare\end{aligned}$$

Next, we will show this $\mathcal{G}$-action is Hamiltonian with curvature map as the moment map.

Lemma 4.9 ([Boa99]). The curvature map

$$\begin{aligned}\Omega : \tilde{\mathcal{A}}_D(\boldsymbol{A}) &\longrightarrow \Omega^2_{2D}\left(\mathbb{P}^1; \operatorname{End} E\right) \\ d - \alpha &\mapsto \Omega(\alpha) := -d\alpha + \alpha \wedge \alpha\end{aligned}$$

is holomorphic, and its tangent map at $\alpha \in \tilde{\mathcal{A}}_D(\boldsymbol{A})$ is

$$\begin{aligned}(d\Omega)_\alpha : T_\alpha\tilde{\mathcal{A}}_D(\boldsymbol{A}) &\longrightarrow \Omega^2\left(\mathbb{P}^1; \operatorname{End} E\right) \\ \phi &\mapsto -d_\alpha\phi\end{aligned}$$

proof. For the second assertion only. By definition:

$$(d\Omega)_\alpha\phi = \frac{d}{dt}\Big|_{t=0} \Omega(\alpha + t\phi) = -d_\alpha\phi \qquad \blacksquare$$

Next, for any $X \in \mathrm{Lie}(\mathcal{G})$, and $\Omega(\alpha) \in \Omega^2(\mathbb{P}^1; \mathrm{End}\, E)$ we can define a paring:

$$\langle \Omega(\alpha), X \rangle := \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{P}^1} \mathrm{Tr}(\Omega(\alpha)X)$$

Theorem 4.4 ([Boa99]). The paring defines the moment map of $\mathcal{G}_1-$action:

$$\mu : \tilde{\mathcal{A}}_D(\boldsymbol{A}) \longrightarrow \mathrm{Lie}(\mathcal{G}_1)^*$$

hence the $\mathcal{G}_1-$action is Hamiltonian, the symplectic quotient is

$$\mu^{-1}(0)/\mathcal{G}_1 = \tilde{\mathcal{A}}_{\mathrm{fl}}(\boldsymbol{A})/\mathcal{G}_1$$

proof. For $X \in \mathrm{Lie}(\mathcal{G})$, we denote

$$\mu_X(\alpha) := \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{P}^1} \mathrm{Tr}(\Omega(\alpha)X) : \tilde{\mathcal{A}}_D(\boldsymbol{A}) \longrightarrow \mathbb{C}$$

we can compute its tangent map $(d\mu_X)_\alpha : W \longrightarrow \mathbb{C}$ by chain rule:

$$\begin{aligned}
(d\mu_X)_\alpha \phi &= \left.\frac{d}{dt}\right|_{t=0} \mu_X(\alpha + t\phi) \\
&= \left.\frac{d}{dt}\right|_{t=0} \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{P}^1} \mathrm{Tr}(\Omega(\alpha + t\phi)X) \\
&= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{P}^1} \mathrm{Tr}\left((d\Omega)_\alpha \phi X\right) \\
&= -\frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{P}^1} \mathrm{Tr}\left((d_\alpha \phi) X\right)
\end{aligned}$$

Notice that, since $X \in \mathrm{Lie}(\mathcal{G}_1)$, $L_i(X) = 0$, hence $\mathrm{Tr}(\phi X)$ is non-singular, thus we have

$$d\mathrm{Tr}(\phi X) = \mathrm{Tr}((d_\alpha \phi)X) - \mathrm{Tr}(\phi \wedge d_\alpha X)$$

applying Stokes formula, we have

$$\begin{aligned}
(d\mu_X)_\alpha \phi &= -\frac{1}{2\pi\sqrt{-1}} \left(\int_{\mathbb{P}^1} d\mathrm{Tr}(\phi X) + \int_{\mathbb{P}^1} \mathrm{Tr}(\phi \wedge d_\alpha X) \right) \\
&= -\frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{P}^1} \mathrm{Tr}(\phi \wedge d_\alpha X) \\
&= \omega_\alpha(\underline{X}, \phi) = -(\iota_{\underline{X}} \omega)_\alpha \phi
\end{aligned}$$

hence μ is indeed a moment map, and the $\mathcal{G}_1-$action is Hamiltonian. ■

Remark 4.8. The larger subgroup $\mathcal{G}_T-$action is also Hamiltonian, but the moment map need to be modified by:

$$\mu : \tilde{\mathcal{A}}_D(\boldsymbol{A}) \longrightarrow \mathrm{Lie}(\mathcal{G}_T)^*$$

where

$$\langle \mu(\alpha), X \rangle := -\sum_{i=1}^{m} \mathrm{Res}_{a_i} L_i(\alpha X) + \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{P}^1} \mathrm{Tr}(\Omega(\alpha) X)$$

5 The Riemann-Hilbert Map is Symplectic

Now, our story can be illustrated in the following diagram:

$$\begin{array}{ccccc} & & \tilde{\mathcal{M}}^0(\mathbf{A}) & \xrightarrow{\tilde{\sigma},\cong} & \tilde{\mathcal{A}}_{\mathrm{fl}}(\mathbf{A})/\mathcal{G}_1 \\ & & {\scriptstyle i}\uparrow & & \downarrow{\scriptstyle \tilde{v},\cong} \\ \tilde{O}_1 \times \cdots \times \tilde{O}_m /\!\!/ GL_n(\mathbb{C}) & \xrightarrow{\tilde{\ell},\cong} & \tilde{\mathcal{M}}^*(\mathbf{A}) & \xrightarrow{\tilde{\nu}} & \tilde{M}_0(\mathbf{A}) \end{array}$$

where i is the inclusion, $\tilde{\nu}$ is the Riemann-Hilbert map defined in Theorem 3.1 and it is injective (by taking Stokes representation induced by a connection ∇). $\tilde{v}$ is the monodromy map defined in Theorem 4.3, and by the Riemann-Hilbert correspondence, it is an isomorphism, $\tilde{\sigma}$ is an isomorphism defined in Theorem 4.2, and $\tilde{\ell}$ appeared in Theorem 2.2.

$\tilde{\mathcal{M}}^*(\boldsymbol{A})$ has the inherited symplectic structure via $\tilde{\ell}$, $\tilde{M}_0(\boldsymbol{A})$ has the inherited symplectic structure from $\tilde{\mathcal{A}}_{\mathrm{fl}}(\boldsymbol{A})/\mathcal{G}_1$, and the symplectic structure in the later moduli space was inherited from the Atiyah-Bott symplectic form on $\tilde{\mathcal{A}}_D(\boldsymbol{A})$, in this chapter, we will show the Riemann-Hilbert map is symplectic, it is equivalent to show:

Theorem 5.1 ([Boa99]). *The map $\hat{\sigma} = \sigma \circ i$:*

$$\hat{\sigma} : \tilde{\mathcal{M}}^*(\boldsymbol{A}) \longrightarrow \tilde{\mathcal{A}}(\boldsymbol{A})$$

is symplectic, where $\sigma : \tilde{\mathcal{M}}^0(\boldsymbol{A}) \longrightarrow \tilde{\mathcal{A}}(\boldsymbol{A})$ was defined in theorem 4.2.

So, the Riemann-Hilbert map:

$$\tilde{\nu} : \tilde{\mathcal{M}}^*(\boldsymbol{A}) \longrightarrow \tilde{M}_0(\boldsymbol{A})$$

is symplectic.

proof. Without the loss of generality, we can assume there is just one pole, i.e $D = k \cdot a$, and the nice formal form near a is A^0, the compatible framing at a is g_0, choose a semi-sphere U which containing a on $\mathbb{P}^1$ such that the coordinate of a is 0, and we can write every connection ∇ under this coordinate chart by

$$\nabla = d - A = d - \left(\frac{A_k}{z^k} + \cdots \frac{A_1}{z}\right)$$

it has no holomorphic parts, since it has no poles on the other semi-sphere, and ∇ is in some equivalent class in $\tilde{\mathcal{M}}^*$ if and only if $A_1 = 0$.

By the construction in Theorem 4.1:

$$\hat{\sigma}(A) = g_A[A]$$

where $g_A \in \mathrm{GL}_n\left(C^\infty\left(\mathbb{P}^1\right)\right)$ satisfying

a). $g_A(a) = g_0$

b). The Taylor expansion of $g_A[A]$ at a is A^0:

$$L_a(g_A[A]) = A^0$$

According to Lemma 2.4, a tangent vector W in $T_\nabla \tilde{\mathcal{M}}^*$ can be write as

$$W = [A, X] + g_0^{-1} \Lambda g_0 := [A, X] + \tilde{\Lambda}$$

where A is the connection matrix of ∇ using the coordinate chart on U, $X \in \mathfrak{g}_k$ with the expression:

$$X = X_0 + X_1 z + \cdots + X_{k-1} z^{k-1}$$

and Λ is actually $\Lambda dz/z \in \mathfrak{t}^*$, notice that W is in fact a matrix-valued meromorphic $(1,0)-$form on $\mathbb{P}^1$.

We know from the proof of Lemma 2.4, a parameterized curve $\gamma(t)$ in $\tilde{\mathcal{M}}^*$ with $\gamma(0) = \nabla$, $\gamma'(0) = W$ can be formulated by

$$\gamma(t) = e^{-tX}\left(A + t\tilde{\Lambda}\right)e^{tX} := A_t$$

hence by Remark 4.6:

$$\hat{\sigma} \circ \gamma(t) = g_{A_t}[A_t]$$

where g_{A_t} is a family of gauge transformations agree with the family of connections A_t, which can be expressed by

$$g_{A_t} = g_A e^{t\tilde{X}}$$

where $\tilde{X} \in \mathrm{End}\left(C^\infty\left(\mathbb{P}^1\right)\right)$, it must satisfy:

a'). $\tilde{X}(0) = X_0$

b'). $L_a\left(g_A e^{t\tilde{X}}[A_t]\right) = A^0$ for every t.

Notice that these two restrains on $\tilde{X}$ are just local conditions, hence we can choose one such that $\tilde{X}$ only supports on a closed disk $\bar{\mathbb{D}}$ containing a, and the Taylor expansion of $\tilde{X}$ at a is precisely X

$$L_a\left(\tilde{X}\right) = X = X_0 + \cdots + X_{k-1} z^{k-1}$$

Let's compute the tangent map $d\hat{\sigma}$:

$$\begin{aligned}
(d\hat{\sigma})_{\nabla} W := \phi &= \frac{\partial}{\partial t}\Big|_{t=0} \left\{ g_A e^{t\tilde{X}}[A_t] \right\} \\
&= \frac{\partial}{\partial t}\Big|_{t=0} \left\{ \left(d g_A e^{t\tilde{X}} \right) e^{-t\tilde{X}} g_A^{-1} + g_A e^{t\tilde{X}} A_t e^{-t\tilde{X}} g_A^{-1} \right\} \\
&= \frac{\partial}{\partial t}\Big|_{t=0} \left\{ (dg_A) g_A^{-1} + g_A e^{t\tilde{X}} \left(t \left(d\tilde{X} \right) + A_t \right) e^{-t\tilde{X}} g_A^{-1} \right\} \\
&= g_A \left(d\tilde{X} + \left[\tilde{X}, A \right] + W \right) g_A^{-1}
\end{aligned}$$

Now, for $W_1, W_2 \in T_{\nabla} \tilde{\mathcal{M}}^*$, where

$$W_j = [A, X_j] + \tilde{\Lambda}_j \qquad j = 1, 2$$

by Lemma 2.5, we have

$$\omega_{\tilde{\mathcal{M}}^*}(W_1, W_2) = \left\langle \tilde{\Lambda}_1, X_2 \right\rangle - \left\langle \tilde{\Lambda}_2, X_1 \right\rangle + \langle A, [X_1, X_2] \rangle$$

Then, by Lemma 4.6:

$$\omega_{\tilde{A}}(\phi_1, \phi_2) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{P}^1} \mathrm{Tr} \left(\left(d\tilde{X}_1 + \left[\tilde{X}_1, A \right] + W_1 \right) \wedge \left(d\tilde{X}_1 + \left[\tilde{X}_2, A \right] + W_2 \right) \right)$$

Notice that $W_1 \wedge W_2$, $W_1 \wedge \left[\tilde{X}_2, A \right]$ and $\left[\tilde{X}_1, A \right] \wedge W_2$ are $(0,2)-$form on $\mathbb{P}^1$ hence zero, thus we have

$$\begin{aligned}
\omega_{\tilde{A}}(\phi_1, \phi_2) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{P}^1} & \mathrm{Tr} \left(d\tilde{X}_1 \wedge d\tilde{X}_2 \right) + \mathrm{Tr} \left(d\tilde{X}_1 \wedge W_2 + W_1 \wedge d\tilde{X}_2 \right) \\
& + \mathrm{Tr} \left(\left[\tilde{X}_1, A \right] \wedge d\tilde{X}_2 \right) + \mathrm{Tr} \left(d\tilde{X}_1 \wedge \left[\tilde{X}_1, A \right] \right)
\end{aligned}$$

by the Poincaré lemma:

$$d\mathrm{Tr} \left(\tilde{X}_1 d\tilde{X}_2 \right) = \mathrm{Tr} \left(d\tilde{X}_1 \wedge d\tilde{X}_2 \right)$$

and $\tilde{X}_1 d\tilde{X}_2$ is non-singular, by Stokes formula:

$$\int_{\mathbb{P}^1} \mathrm{Tr} \left(d\tilde{X}_1 \wedge d\tilde{X}_2 \right) = \int_{\mathbb{P}^1} d\mathrm{Tr} \left(\tilde{X}_1 d\tilde{X}_2 \right) = 0$$

and since $\tilde{X}_j$ was chose only supported on $\bar{\mathbb{D}}$, hence we have

$$\begin{aligned}
\omega_{\tilde{A}}(\phi_1, \phi_2) = \frac{1}{2\pi\sqrt{-1}} \int_{\bar{\mathbb{D}}} & \mathrm{Tr} \left(d\tilde{X}_1 \wedge W_2 + W_1 \wedge d\tilde{X}_2 \right) \\
& + \mathrm{Tr} \left(\left[\tilde{X}_1, A \right] \wedge d\tilde{X}_2 \right) + \mathrm{Tr} \left(d\tilde{X}_1 \wedge \left[\tilde{X}_2, A \right] \right)
\end{aligned}$$

Next, observe that

$$\mathrm{Tr}\left(\left[\tilde{X}_1, A\right] \wedge d\tilde{X}_2\right) = d\mathrm{Tr}\left(\left[\tilde{X}_1, A\right] \tilde{X}_2\right) - \mathrm{Tr}\left(\left(d\left[\tilde{X}_1, A\right]\right) \cdot \tilde{X}_2\right)$$

and

$$\left[\tilde{X}_1, A\right] \wedge \tilde{X}_2 = \left[\tilde{X}_1, A\right] \cdot \frac{\partial \tilde{X}_2}{\partial \bar{z}} dz \wedge d\bar{z}$$

$$\left(d\left[\tilde{X}_1, A\right]\right) \tilde{X}_2 = -\left[\frac{\partial \tilde{X}_1}{\partial \bar{z}}, A\right] \cdot \tilde{X}_2 dz \wedge d\bar{z}$$

hence

$$\begin{aligned} I &:= \int_{\bar{\mathbb{D}}} \mathrm{Tr}\left(\left[\tilde{X}_1, A\right] \wedge d\tilde{X}_2\right) \\ &= \int_{\bar{\mathbb{D}}} \mathrm{Tr}\left(\left[\tilde{X}_1, A\right] \cdot \frac{\partial \tilde{X}_2}{\partial \bar{z}}\right) dz \wedge d\bar{z} \\ &= \int_{\bar{\mathbb{D}}} d\mathrm{Tr}\left(\left[\tilde{X}_1, A\right] \tilde{X}_2\right) - \int_{\bar{\mathbb{D}}} \mathrm{Tr}\left(\left(d\left[\tilde{X}_1, A\right]\right) \cdot \tilde{X}_2\right) \\ &= \int_{\partial \bar{D}} \mathrm{Tr}\left(\left[\tilde{X}_1, A\right] \tilde{X}_2\right) + \int_{\bar{\mathbb{D}}} \mathrm{Tr}\left(\left[\frac{\partial \tilde{X}_1}{\partial \bar{z}}, A\right] \cdot \tilde{X}_2\right) dz \wedge d\bar{z} \end{aligned}$$

recall that by Remark 2.1 item (3), the paring is given by

$$\begin{aligned} \langle A, \tilde{X}_j \rangle := \mathrm{Res}_a\left(\mathrm{Tr}\left(A\tilde{X}_j\right)\right) &= \mathrm{Res}_a\left(\mathrm{Tr}\left(A \cdot L_a\left(\tilde{X}_j\right)\right)\right) \\ &= \mathrm{Res}_a\left(\mathrm{Tr}\left(A \cdot X_j\right)\right) \end{aligned}$$

thus by residue theorem we have

$$\begin{aligned} \int_{\partial \bar{D}} \mathrm{Tr}\left(\left[\tilde{X}_1, A\right] \tilde{X}_2\right) &= 2\pi\sqrt{-1} \cdot \mathrm{Res}_a\left(\mathrm{Tr}([\tilde{X}_1, A] \cdot \tilde{X}_2)\right) \\ &= 2\pi\sqrt{-1} \cdot \mathrm{Res}_a(\mathrm{Tr}([X_1, A] \cdot X_2)) \\ &= 2\pi\sqrt{-1} \cdot \langle A, [X_1, X_2] \rangle \end{aligned}$$

to compute the second part of I, we again use the integration by parts:

$$\begin{aligned} \int_{\bar{\mathbb{D}}} \mathrm{Tr}\left(\left[\frac{\partial \tilde{X}_1}{\partial \bar{z}}, A\right] \cdot \tilde{X}_2\right) dz \wedge d\bar{z} &= \int_{\bar{\mathbb{D}}} \frac{\partial}{\partial \bar{z}} \mathrm{Tr}\left(\left[\tilde{X}_1, A\right] \cdot \tilde{X}_2\right) dz \wedge d\bar{z} - I \\ &= -I \end{aligned}$$

where the last step comes from the *enhanced Cauchy formula* [GH14]:

$$\int_D \frac{\partial f}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{(z-a)^k} = \frac{2\pi\sqrt{-1}}{(k-1)!} \cdot \frac{\partial^{k-1} f}{\partial z^{k-1}}(a) - \int_{\partial D} \frac{f(z)dz}{(z-a)^k}$$

hence

$$I = \frac{1}{2}\langle A, [X_1, X_2]\rangle$$

so now, our symplectic form becomes to

$$\begin{aligned}
\omega_{\tilde{\mathcal{A}}}(\phi_1, \phi_2) &= \frac{1}{2\pi\sqrt{-1}}\int_{\bar{\mathbb{D}}} \mathrm{Tr}\left(d\tilde{X}_1 \wedge W_2 + W_1 \wedge d\tilde{X}_2\right) + \langle A, [X_1, X_2]\rangle \\
&= \frac{1}{2\pi\sqrt{-1}}\int_{\bar{\mathbb{D}}} \mathrm{Tr}\left(d\tilde{X}_1 \wedge [A, X_2]\right) + \mathrm{Tr}([A, X_1] \wedge d\tilde{X}_2) \\
&+ \frac{1}{2\pi\sqrt{-1}}\int_{\bar{\mathbb{D}}} \mathrm{Tr}\left(d\tilde{X}_1 \wedge \tilde{\Lambda}_2 + \tilde{\Lambda}_1 \wedge d\tilde{X}_2\right) + \langle A, [X_1, X_2]\rangle
\end{aligned}$$

notice that $\partial X_j/\partial\bar{z} = 0$, hence we can use a similar method to compute

$$\begin{aligned}
J &:= \int_{\bar{\mathbb{D}}} \mathrm{Tr}\left(d\tilde{X}_1 \wedge [A, X_2]\right) = \int_{\bar{\mathbb{D}}} \mathrm{Tr}\left(\frac{\partial\tilde{X}_1}{\partial\bar{z}} \cdot [X_2, A]\right) dz \wedge d\bar{z} \\
&= \int_{\bar{\mathbb{D}}} \frac{\partial}{\partial\bar{z}} \mathrm{Tr}\left(\tilde{X}_1 \cdot [X_2, A]\right) dz \wedge d\bar{z} \\
&= \int_{\bar{\mathbb{D}}} d\mathrm{Tr}\left(\tilde{X}_1 \cdot [X_2, A]\right) = \int_{\partial\bar{\mathbb{D}}} \mathrm{Tr}\left(\tilde{X}_1 \cdot [X_2, A]\right) \\
&= 2\pi\sqrt{-1} \cdot \mathrm{Res}_a(\mathrm{Tr}(X_1 \cdot [X_2, A])) = -2\pi\sqrt{-1} \cdot \langle A, [X_1, X_2]\rangle
\end{aligned}$$

so now, our last mission is to compute

$$\omega_{\tilde{\mathcal{A}}}(\phi_1, \phi_2) = -\langle A, [X_1, X_2]\rangle + \frac{1}{2\pi\sqrt{-1}}\int_{\bar{\mathbb{D}}} \mathrm{Tr}\left(d\tilde{X}_1 \wedge \tilde{\Lambda}_2 + \tilde{\Lambda}_1 \wedge d\tilde{X}_2\right)$$

since Λ_j are $(1,0)-$forms, again, applying Poicaré lemma and integration by parts, we obtain:

$$\begin{aligned}
\int_{\bar{\mathbb{D}}} \mathrm{Tr}\left(\tilde{\Lambda}_1 \wedge d\tilde{X}_2\right) &= \int_{\bar{\mathbb{D}}} d\mathrm{Tr}\left(\tilde{\Lambda}_1 \cdot \tilde{X}_2\right) \\
&= \int_{\partial\bar{\mathbb{D}}} \mathrm{Tr}\left(\tilde{X}_2 \cdot \tilde{\Lambda}_1\right) = \mathrm{Res}_a \mathrm{Tr}\left(X_2 \tilde{\Lambda}_1\right) \\
&= \left\langle \tilde{\Lambda}_1, X_2 \right\rangle
\end{aligned}$$

to sum up, we have:

$$\omega_{\tilde{\mathcal{A}}}\left((d\hat{\sigma})_\nabla W_1, (d\hat{\sigma})_\nabla W_2\right) = \omega_{\tilde{\mathcal{M}}^*}(W_1, W_2)$$

hence $\hat{\sigma}$ is symplectic, as was to be shown. ■

6 Further Prospects

6.1 The Symplectic Geometry of Isomonodromic Deformation Equations

6.1.1 The Riemann Isomonodromy Problem

The general isomonodromic deformation equations, introduced by Jimbo, Miwa and Ueno [JUM80], was originated from the Schlesinger euqations [Sch12].

Let's consider a Fuchsian system on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$:

$$d\boldsymbol{y} = A\boldsymbol{y}$$

A is a matrix of meromorphic 1-forms with only simple poles $a_1, ..., a_m$. If we assume ∞ is not a pole of A, then we can write A as

$$A = \sum_{i=1}^{n} \frac{A_i}{z - a_i}$$

where $A_i \in \mathfrak{gl}_n^*(\mathbb{C})$. It induces a monodromy representation of the fundamental group:

$$\rho : \pi_1 \left(\mathbb{P}^1 \setminus \{a_1, ..., a_m\}\right) \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

now, the isomonodromic deformation problem is, if we assume the poles $a_1, ..., a_m$ are varying smoothly on $\mathbb{C}$, what conditions shall those A_i have such that they induce the same monodromy representation? In other words, there are many different Fuchsian systems with the same monodromy behavior, how to determine all such systems? Schlesinger in his 1912 paper [Sch12] claimed that all such systems are governed by the equations which are nowadays called the *Schlesinger equations*:

$$\begin{cases} \frac{\partial A_i}{\partial a_j} = \frac{[A_i, A_j]}{a_i - a_j} & i \neq j \\ \frac{\partial A_i}{\partial a_i} = -\sum_{j \neq i} \frac{[A_i, A_j]}{a_i - a_j} & i = j \end{cases}$$

Now, if the equation is no more Fuchsian, i.e, the orders of the poles are greater than 2, the "monodromy" data will be more complicated, the Stokes matrices should be involved.

Let's consider an (irregular) equation with poles at $a_1, ..., a_m$, the order of each pole a_i is n_i, and has no further pole at ∞, we can write such an equation as

$$d\boldsymbol{y} = A\boldsymbol{y} = \sum_{i=1}^{m} \left(\frac{{}^iA_{n_i}}{(z - a_i)^{n_i}} + \cdots + \frac{{}^iA_1}{(z - a_i)} \right) \boldsymbol{y}$$

We need to request first that this equation is generic, i.e, the matrices ${}^iA_{n_i}$ are diagonalisable and with distinct eigenvalues, and second, this equation is formal equivalent to a diagonal system, that is to say there exists a formal gauge transformation $\hat{F}_i \in \mathrm{GL}_n(\mathbb{C}[\![z]\!])$ for each pole a_i such that

$$\hat{F}_i\left[{}^iA^0\right] = {}^iA$$

where ${}^iA^0$ is a diagonal matrix. Thus this will determine a group of Stokes matrices $({}^1\boldsymbol{S}, ..., {}^m\boldsymbol{S})$ near each pole a_i, where

$${}^i\boldsymbol{S} = \left({}^iS_1, ..., {}^iS_{r_i}\right)$$

is the Stokes matrices at the pole a_i, r_i is cardinality of the anti-Stokes directions at a_i.

Now, let $a_1, .., a_m$ vary smoothly on $\mathbb{C}$, the isomonodromic deformation problem aims to find all irregular systems with the following "monodromy data" being fixed:

a). the number and the order of the poles;

b). the properties of being generic and formal diagonalisable;

c). the monodromy representation of the fundamental group (the exponents of formal monodromy);

d). the Stokes matrices near each pole.

Let $X_k(a)$ be the set of all irregular types at $a \in \mathbb{P}^1$ of order k, i.e

$$X_k(a) = \left\{ dQ = A_k^0 \frac{dz}{z^k} + \cdots + A_2^0 \frac{dz}{z^2} \middle| A_i^0 \in \mathfrak{t} \right\}$$

The data that is deforming now is:

i). The position of the poles $a_1, ..., a_m \in \mathbb{P}^1$;

ii). The irregular types (the residue parts are fixed by c), since they are the exponents of formal monodromy):

$$\boldsymbol{A} = \left({}^1A^0, ..., {}^mA^0\right)$$

We can define the manifold of the deformation data:

Definition 6.1 ([Boa99]). The Jimbo-Miwa-Ueno deformation manifold is defined by

$$X = \left\{(\boldsymbol{a}, \boldsymbol{A}) := t | \boldsymbol{a} = (a_1, ..., a_m) \in \mathbb{P}^1, \boldsymbol{A} = \left({}^1A^0, ..., {}^mA^0\right), {}^iA^0 \in X_{k_i}(a_i)\right\}$$

Define a 1-form on X by:

$$\Omega := \sum_{i=1}^{m} \Big(A(z,t) da_i - \hat{F}_i(z,t) D \left({}^iA_1^0 + \cdots + {}^iA_{n_i}^0\right) \hat{F}_i^{-1}(z,t) \Big)$$

where D is the exterior differential with respect to the components of ${}^iA_k^0$ only, and we use $A(z,t), \hat{F}(z,t)$ to indicate that they are depending on the deformation parameter t.

Jimbo, Miwa and Ueno's answer to the isomonodromic problem in the general case is:

Theorem 6.1 ([JUM80]). The linear ODEs with the monodromy data a),b),c) and d) fixed are determined by the following equation:

$$dA + [\Omega, A] + \frac{d\Omega}{dz} = 0$$

This equation is called the Jimbo-Miwa-Ueno isomonodromic deformation equation, it is a non-linear differential equation satisfying the Painlevé property [Con12].

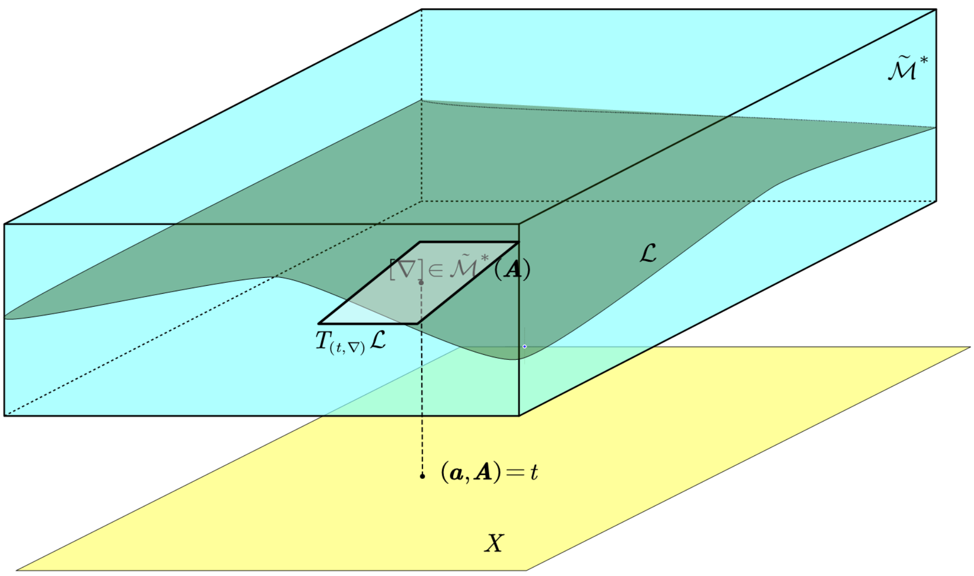
6.1.2 Isomonodromic Connection

Now, back to our moduli space $\tilde{\mathcal{M}}^*(\boldsymbol{A})$, we can associate a fibre $\tilde{\mathcal{M}}^*(\boldsymbol{A})$ at each point $t = (\boldsymbol{a}, \boldsymbol{A}) \in X$, hence we can define a fibre bundle of moduli spaces, denoted by $\tilde{\mathcal{M}}^*$, the projection is simply

$$(t, (E, \nabla, \boldsymbol{g})) \mapsto t$$

Boalch showed that:

Lemma 6.1 ([Boa99]). The bundle $\tilde{\mathcal{M}}^*$ of extended moduli spaces is a complex manifold, moreover, the projection defined above makes it a symplectic fibre bundle.



the horizontal distribution on $\tilde{\mathcal{M}}^*$

The isomonodromic deformation equation canonically determines an Ehressmann connection on this fibre bundle. In fact, choose any $(t, \nabla_t) \in \tilde{\mathcal{M}}^*$, the set of the solutions of the isomonodromic deformation equation:

$$\left\{ \nabla = d - A | dA + [\Omega, A] + \frac{d\Omega}{dz}, A(t) = \nabla_t \right\} := \mathcal{L}_{\nabla_t}$$

i.e the collection with the monodromy data as ∇_t, is the submanifold of $\tilde{\mathcal{M}}^*$, its dimension equals to $\dim X$, the horizontal distribution can be given by

$$\Phi : (t, \nabla_t) \mapsto T_{\nabla_t}\mathcal{L}_{\nabla_t}$$

This connection is called the *isomonodromic connection.* In [Boa99], Boalch gave the symplectic nature of the Jimbo-Miwa-Ueno's isomonodromic deformation equation:

Theorem 6.2 ([Boa99]). The isomonodromic connection on the fibre bundle $\tilde{\mathcal{M}}^*$ of extended moduli spaces is a flat symplectic connection, i.e, the local analytic diffeomorphisms induced by the isomonodromic connection between the fibres of $\tilde{\mathcal{M}}^*$ are symplectic diffeomorphisms.

6.2 The Computations of Stokes Matrices

As we mentioned before, the Stokes matrices are very important invariants in the theory of linear ODEs, thus it is our natural mission to compute them. However, even in the case of 2nd order, the computations of them are rather complicated. For example, Prof. Xiaomeng Xu gave an explicit formula of the Stokes matrices of the following ODE [Xu16]:

$$\frac{d\boldsymbol{y}}{dz} = \left(\frac{\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}}{z^2} + \frac{\begin{pmatrix} u_1 & a \\ b & u_2 \end{pmatrix}}{z} \right) \boldsymbol{y}$$

although this is a very simple case, the computation is already very complicated.

The difficulty is, when we do the re-summation process of a formal solution $\hat{Y}$, for example, the Borel-Laplace transformation, there will be singularities appeared after doing the Borel transformation, if the singularities are poles, the Stokes matrices can be computed by the residue formula, however, in the most of the cases, the singularities are even worse, they will be essential singularities, hence the residue formula cannot show its power any more. In the late 20th century, the French mathematician J.Écalle developed a method called the *Alien calculus* [SM10], which are now turned out to be a powerful tool to deal with the essential singularities.

There are also some algebraic ways to compute the Stokes matrices. In [dHMS20], the authors used Kashiwara's Riemann-Hilbert correspondence for holonomic $\mathcal{D}-$modules to compute the Stokes matrices attached to irregular singularities arises from Fourier-Laplace transforms of regular systems, and then in [HJ22], the authors used the similar method to compute the Stokes matrices of the generalised Airy's equations.

6.3 A Glance at Nonabelian Hodge Theory

When a compact Riemann surface X is given, one can consider two different but equivalent things, the first one is the irreducible representation of the fundamental group $\pi_1(X) \longrightarrow \mathrm{GL_n}(\mathbb{C})$, the other one is the (flat) holomorphic connections on holomorphic degree 0 vector bundles. The Riemann-Hilbert correspondence on Riemann surfaces asserts that there is a 1-1 correspondence between the holomorphic connections and the irreducible representations of the fundamental groups (by taking holonomy representations), if we use the terminologies in moduli spaces, this correspondence is saying that there is an isomorphism between the moduli spaces:

$$\mathcal{M}_{\mathrm{dR}}(X,n) \cong \mathcal{M}_B(X,n) := \mathrm{Hom}_{\mathrm{irr}}(\pi_1(\mathbb{C}); \mathrm{GL_n}(\mathbb{C}))/\mathrm{GL_n}(\mathbb{C})$$

where $\mathcal{M}_{\mathrm{dR}}(X,n)$ on the left hand side is the moduli space of holomorphic connections on rank n degree 0 vector bundles $E \longrightarrow X$, it will be called the **de Rham site** in our story, and $\mathcal{M}_{\mathrm{B}}(X,n)$, the moduli space of representations, will be called the **Betti site**.

In [NS65], Narasimhan and Seshadri found the equivalence between the moduli space of stable vector bundles $\mathcal{U}_n(X)$ and the moduli space of irreducible unitary representations of $\pi_1(X)$. Later, Hitchin, Simpson and some other mathematicians found that it will be better to consider the "complex version", then the notion of Higgs bundles yields.

A Higgs bundle (E, Φ) is just a holomorphic vector bundle E together with an extra Higgs field $\Phi \in \Gamma(\Omega^1 \otimes \mathrm{End}(E))$, we will use $\mathcal{M}_{\mathrm{Dol}}(X,n)$ to denote the moduli space of rank n degree 0 Higgs bundles, this will become the 3rd site of our story, namely, the **Dolbeault Site**. These 3 sites are isomorphic [Sim92]

$$\mathcal{M}_{\mathrm{Dol}}(X,n) \cong \mathcal{M}_{\mathrm{dR}}(X,n) \cong \mathcal{M}_B(X,n)$$

In Hodge theory, we have Hodge decomposition for any compact Kähler manifold X [Voi03]:

$$H^k(X;\mathbb{C}) \cong \bigoplus_{p+q=k} H^{(p,q)}(X;\mathbb{C})$$

when we take $k = 1$:

$$H^1(X;\mathbb{C}) = H^{(1,0)}(X;\mathbb{C}) \oplus H^{(0,1)}(X;\mathbb{C})$$

applying Huerwitz theorem and Dolbeault theorem [Huy05], we will have:

$$\mathrm{Hom}(\pi_1(X);\mathbb{C}) \cong H^0(X;\Omega^1) \oplus H^1(X;\mathcal{O}_X)$$

Recall that the holomorphic vector bundles can be classified by the 1st cohomology group of non-Abelian sheaf $H^1(X, \mathcal{GL}_n)$ (the cocycles are exactly the transition functions).

Hence a Higgs bundle (E, Φ) can be identified with an element in $H^0(X; \mathrm{End} \otimes \Omega^1) \oplus H^1(X, \mathcal{GL}_n)$, which is a non-Abelian analogue of the right hand side in Hodge theory, and by replacing $\mathbb{C}$ to a non-Abelian group $\mathrm{GL_n}(\mathbb{C})$, the left hand side becomes the Betti site. So, the correspondence between Dolbeault sites and Betti sites is a non-Abelian analogue of Hodge theory.

So now, we can aware that what we've done so far is a generalisation on the de Rham site, by replacing the holomorphic connections to the meromorphic ones, and the Betti site is no longer the representations of the fundamental group, but a groupoid. In Boalch's later work [Boa13] [BB04], he established a meromorphic version of the correspondence in those 3 different sites, and called it the "wild non-Abelian Hodge theory".

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