Notes on Seiberg-Witten Gauge Theory

Zhiyuan Liu

April 30, 2024

Abstract

This is a learning note for Seiberg-Witten gauge theory. The main part of this note will include the classical theory of Seiberg-Witten invariants and their applications in the topology of smooth 3,4-manifolds. For some excellent learning literatures, I'd like to refer to [1–3]. This note will be innovated inconstantly, the later topics will contain the theory and applications of generalised Seiberg-Witten equations.

In order to define the Seiberg-Witten equations, we need to establish what is a Spin structure at first, since Seiberg-Witten equations describe the existence of some certain *spinor fields* ψ and some connections A on a line bundle associated to a chosen Spin^c structure.

Like other gauge theories (Yang-Mills, Chern-Simons, etc.), the solutions of Seiberg-Witten equations are gauge invariant under a gauge group $\mathcal{G}$ action, the speciality in here is that the gauge group $\mathcal{G}$ is Abelian (a group of $\mathrm{U}(1)-$valued smooth functions), that's why the Seiberg-Witten gauge theory is also called an *Abelian gauge theory*. Then it is natural to discuss the *Seiberg-Witten moduli space* $\mathcal{M}_{SW}$, that is the space of solutions modulo the gauge equivalency.

However, the study of the moduli spaces is a hard part in gauge theory, since the moduli spaces maybe not compact, not smooth or hard to give an orientation. However, different from other gauge theories (especially for Yang-Mills), the Seiberg-Witten moduli spaces have a lot of good properties due to the less non-linearity of the Seiberg-Witten equations, which are 1st order non-linear elliptic PDEs.

The main good properties of the Seiberg-Witten moduli space $\mathcal{M}_{SW}$ are: it is compact, it is smooth under a small perturbation, it is orientated so that we can do intersections (and hence we can define numerical invariants, namely the celebrated Seiberg-Witten invariants).

The proof of these properties are standard arguments in geometrical analysis and infinite-dimensional differential topology (mainly the Fredholm theory), and these techniques are frequently appeared in many other geometric theories which involve the analysis of elliptic PDEs (such as Floer homology, $J-$holomorphic curves and Gromov invariants, Donaldson theory, etc.). At this point, the proof of these properties will be a core part through this note, because they are toy models which can help us to understand more difficult topics.

Since the analytic nature of Seiberg-Witten gauge theory is much easier than Yang-Mills, and it can study the topology of low-dimensional manifolds as well, it makes learning Seiberg-Witten theory a good start point for those who want to know the topological applications of gauge theory. I will write some of its applications in the topology of Kähler surfaces. For more applications of gauge theory to complex geometry and $4-$manifolds, I heard that [2, 4, 5] are excellent references.

Contents

Part I: Spin Geometry

1 Clifford Algebras and Spin Groups

Let's first start with some algebraic preliminaries, a good reference of this part can be found in [2, 6]. A little preliminary knowledge about quaternion is expected, they can be found in Appendix 1.A.

The notion of Clifford algebras arises from the algebraic presentations of rotations.

For example (cf. Proposition 1.1), the rotations in $\mathbb{R}^2$ can be presented by multiplications of unit complex numbers; the rotations in $\mathbb{R}^3$ can be presented by unit quaternions via $v \mapsto hv\bar{h}$, where $h \in \mathbb{H}$ with $|h| = 1$, and $v \in \mathbb{R}^3$ is identified with $\mathrm{Im}\mathbb{H} = \mathbb{R}^3$; as for $\mathbb{R}^4$ the presentations of rotations need 2 unit quaternions via $h \mapsto q_+ h q_-$, where $q_\pm \in \mathrm{U}(1, \mathbb{H})$, and $h \in \mathbb{R}^4$ is identified with $\mathbb{H}$.

We shall see later that these $\mathbb{C}, \mathbb{H}$ or $\mathbb{H} \oplus \mathbb{H}$ are actually the irreducible representation spaces of the Clifford algebras associated to the underlying Euclidean spaces. Their elements are called *spinors*, which can be vaguely understood, as was implied by its literally meaning, the generators of rotations.

1.1 Clifford Algebras and Their Representations

1.1.1 Clifford Algebras and Their Complexifications

Let $(V, \langle \cdot, \cdot \rangle)$ be a real $n-$dimensional vector space equipped with an inner product.

Definition 1.1 (Clifford Algebras)**.** The Clifford algebra $\mathrm{C}\ell(V)$ of V is defined by:

$$\mathrm{C}\ell(V) := \frac{TV}{\langle v \otimes v - |v|^2 |\, v \in V \rangle}$$

where TV is the tensor algebra of V, $|v|^2$ means $\langle v, v \rangle$ scalar products with the identity $1 \in TV$. ♣

The multiplication in $\mathrm{C}\ell(V)$ is denoted by a dot $\cdot$. Briefly speaking, the Clifford algebra of V is just an algebra generated by elements in V subordinate to the relation $v \cdot v = -|v|^2$. Also, notice that for any $x, y \in V$, we have

$$\begin{aligned} (x+y)^2 := (x+y) \cdot (x+y) &= -\left(|x|^2 + |y|^2\right) + x \cdot y + y \cdot x \\ &= -\langle x+y, x+y \rangle = -\left(|x|^2 + |y|^2\right) - 2\langle x, y \rangle \end{aligned}$$

we see that

$$x \cdot y + y \cdot x = -2\langle x, y \rangle \tag{1}$$

In particular one can deduce that

Lemma 1.1. *If $x_1, ..., x_n$ is an orthonormal basis of V, then $C\ell(V)$ is an algebra generated by $1, x_1, ..., x_n$ subordinate to the relation:*

$$\begin{cases} x_i^2 = -1, \\ x_i \cdot x_j = x_j \cdot x_i,\ i \neq j \end{cases} \Longleftrightarrow x_i x_j + x_j x_i = -2\delta_{ij}$$

Notice that the Clifford algebras are algebras over $\mathbb{R}$, thus it is convenient to define its complexification by tensor product $C\ell(V) \otimes_{\mathbb{R}} \mathbb{C}$. Now let's compute some examples by lemma 1.1.

Example 1.1. (1) If $V = \mathbb{R}$, then $C\ell(\mathbb{R}) = \mathbb{R}[x]/\langle x^2 + 1\rangle$, which is $\mathbb{C}$, hence the complexification algebra is $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^2$.[1]

(2) If $V = \mathbb{R}^2$, then $C\ell(\mathbb{R}^2)$ is generated by $1, x, y$, satisfying

$$x^2 = y^2 = -1, \quad xy = -yx$$

note that if we denoted by i, j, k the imaginary unit of the quaternions $\mathbb{H}$, then by letting $x = i, y = j$, and $xy = k$, then we find that $C\ell(\mathbb{R}^2) \cong \mathbb{H}$.

To compute the complexification algebra, we recall that each quaternion has a matrix presentation by 2×2 complex matrices (see Appendix 1.A), hence $\mathbb{H}$ can be viewed as a subalgebra of $\mathrm{End}(\mathbb{C} \oplus \mathbb{C})$, therefore

$$C\ell(\mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{End}(\mathbb{C} \oplus \mathbb{C})$$

(3) If $V = \mathbb{R}^3$, then $C\ell(\mathbb{R}^3)$ is generated by $1, x, y, z$ satisfying

$$\begin{cases} x^2 = y^2 = z^2 = -1 \\ xy = -yx, xz = -zx, yz = -zy \end{cases} \tag{2}$$

We claim that there is an $\mathbb{R}-$algebra isomorphism $C\ell(\mathbb{R}^3) \cong \mathbb{H} \oplus \mathbb{H}$.

One should be careful that an element $(h_1, h_2) \in \mathbb{H} \oplus \mathbb{H}$ is identified with the diagonal matrix $\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$, hence the multiplication is defined by the matrix multiplication after a diagonal embedding. Therefore, as an $\mathbb{R}-$algebra, $\mathbb{H} \oplus \mathbb{H}$ is generated by the matrices

$$e_1 = \begin{pmatrix} -k & \\ & k \end{pmatrix}, e_2 = \begin{pmatrix} -j & \\ & j \end{pmatrix}, e_3 = \begin{pmatrix} -i & \\ & i \end{pmatrix}$$

and the generators also satisfy the relation (2).

[1]Note that $\mathbb{C}$ as an $\mathbb{R}-$algebra is actually $\mathbb{R} \oplus \mathbb{R}$, hence the complexification is simply by replacing $\mathbb{R}$ to $\mathbb{C}$ in each component.

With this in mind, we can define the isomorphism by expanding $x \mapsto e_1$, $y \mapsto e_2$ and $z \mapsto e_3$. [2]

By the computation in 1.1(2), it is clearly that the complexification algebra is simply $\mathrm{End}(\mathbb{C}^2) \oplus \mathrm{End}(\mathbb{C}^2)$.

(4) By the same method, one can check that $C\ell(\mathbb{R}^4) \cong \mathrm{End}(\mathbb{H} \oplus \mathbb{H})$ (the multiplication in the later algebra is the matrix multiplication). Indeed, the algebra $\mathrm{End}(\mathbb{H} \oplus \mathbb{H})$ is generated by

$$\begin{aligned} e_1 &= \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, e_2 = \begin{pmatrix} & i \\ i & \end{pmatrix}, e_3 = \begin{pmatrix} & j \\ j & \end{pmatrix}, e_4 = \begin{pmatrix} & k \\ k & \end{pmatrix} \\ \epsilon_1 &= \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \epsilon_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \epsilon_3 = \begin{pmatrix} & -j \\ j & \end{pmatrix}, \epsilon_4 = \begin{pmatrix} & -k \\ k & \end{pmatrix} \end{aligned} \tag{3}$$

the isomorphism is defined by identifying $x_1 \dots x_4$ with $e_1 \dots e_4$ and $x_2x_3x_4 \dots x_1x_2x_3$ with $\epsilon_1 \dots \epsilon_4$ correspondingly.

Consequently, the complexified Clifford algebra is just by replacing each entry of $A \in \mathrm{End}(\mathbb{H} \oplus \mathbb{H})$ to a 2×2 complex matrix, which is $\mathrm{End}(\mathbb{C}^4)$.

(5) In fact, the Clifford algebras have a periodicity 8:

$$C\ell\left(\mathbb{R}^{k+8}\right) \cong C\ell\left(\mathbb{R}^k\right)$$

therefore it's enough to calculate all Clifford algebras for $k \leq 8$. The computations and the proof of the above periodicity can be found in [6, §1.4], the list of the classification can be found in table 1. ♣

Remark 1.1. As was illustrated in Example 1.1 (4), we can associate each vector $x \in \mathbb{R}^4$ with a 4×4 complex matrix presentation. That is by replacing i, j, k of those e_i's in (3) to some 2×2 complex matrices defined in (13). ♣

1.1.2 Representations and Clifford Modules

Definition 1.2 (Representation). A (complex) representation of a Clifford algebra $C\ell(V)$ is an $\mathbb{R}-$algebra homomorphism:

$$\rho : C\ell(V) \longrightarrow \mathrm{End}_{\mathbb{C}}(W)$$

where W is a $\mathbb{C}-$linear space, and $\mathrm{End}_{\mathbb{C}}(W)$ is viewed as an $\mathbb{R}-$algebra. ♣

[2] On the reverse side, the isomorphism is defined by expanding

$$\begin{aligned} (1,0) \mapsto \frac{1+xyz}{2}, (0,1) \mapsto \frac{1-xyz}{2}, (i,0) \mapsto \frac{xy-z}{2}, (0,i) \mapsto \frac{xy+z}{2} \\ (j,0) \mapsto \frac{xz-y}{2}, (0,j) \mapsto \frac{xz+y}{2}, (k,0) \mapsto \frac{yz-x}{2}, (0,k) \mapsto \frac{yz+x}{2} \end{aligned}$$

Remark 1.2. If W is a representation space of a Clifford algebra $\mathrm{C}\ell(V)$, definition 1.2 is equivalent to say that W endows with an $V-$action, called the *Clifford multiplication*:

$$\mathrm{C}\ell : V \otimes W \longrightarrow W,\ v \otimes w \mapsto v \cdot w$$

which satisfying

$$v \cdot (v \cdot w) = -|v|^2 w$$

The representation space W of $\mathrm{C}\ell(V)$ is also called a *Clifford module*. ♣

Here are several examples of Clifford modules.

Example 1.2. (1) If W is a quaternionic vector space, in particular it is also a vector space over $\mathbb{C}$, then W is a $\mathrm{C}\ell(\mathbb{R}^3)-$module. Because for a $q \in \mathbb{R}^3 \cong \operatorname{Im}\mathbb{H}$ one always has $q^2 = -|q|^2$.

In particular $\mathbb{H}$ is a $\mathrm{C}\ell(\mathbb{R}^3)-$module, the Clifford multiplication is simply the quaternion multiplication.

As a notation convention, we denoted by $\$$ the quaternion $\mathbb{H}$ when it's regarded as a *vector space* over $\mathbb{C}$, and $\mathrm{End}(\$)$ stands for the *complex* endomorphisms of $\$$.

(2) If W_1, W_2 are two quaternionic vector spaces, then $W_1 \oplus W_2$ is a $\mathrm{C}\ell(\mathbb{R}^4)-$module. Here the Clifford multiplication is defined by (identifying $\mathbb{R}^4$ with $\mathbb{H}$)

$$\begin{aligned} \mathrm{C}\ell : \mathbb{R}^4 \otimes (W_1 \oplus W_2) &\longrightarrow W_1 \oplus W_2 \\ h \otimes (w_1, w_2) &\mapsto (w_1, w_2) \begin{pmatrix} 0 & -\bar{h} \\ h & 0 \end{pmatrix} \end{aligned} \tag{4}$$

since one can easily check that

$$\begin{pmatrix} & -\bar{h} \\ h & \end{pmatrix}^2 = -|h|^2 \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

In particular, $\mathbb{H} \oplus \mathbb{H}$ is a $\mathrm{C}\ell(\mathbb{R}^4)-$module, which is denoted by $\$_+ \oplus \$_-$ by the previously convention.

(3) For any Euclidean space V, the complexified exterior algebra

$$\bigwedge V^*_{\mathbb{C}} := \left(\bigoplus_{k \geq 0} \bigwedge^k V^* \right) \otimes_{\mathbb{R}} \mathbb{C}$$

is a $\mathrm{C}\ell(V)-$module. The Clifford multiplication is given by [3]

$$\mathrm{C}\ell : v \otimes \omega \mapsto \iota_v \omega - \langle v, \cdot \rangle \wedge \omega \tag{5}$$

[3] By identifying $V \cong V^*$ via Euclidean inner product, $\bigwedge V_{\mathbb{C}}$ is also a $\mathrm{C}\ell(V)-$module, where the Clifford multiplication can also be write by (5): $v \cdot w := \iota_v w - v \wedge w$, where $w \in \bigwedge V_{\mathbb{C}}$. One should notice that, if we write $w = w_1 \wedge \cdots \wedge w_k$, the contraction $\iota_v w$ is actually

$$\iota_v (w_1 \wedge \cdots \wedge w_k) := \sum_{i=1}^{k} (-1)^{i-1} \langle v, w_i \rangle w_1 \wedge \cdots \wedge \hat{w}_i \wedge \cdots \wedge w_k$$

where $v \in V, \omega \in \bigwedge V_{\mathbb{C}}^*$. In fact,

$$\begin{aligned} v \cdot (v \cdot \omega) &= \iota_v(\iota_v \omega - \langle v, \cdot \rangle \wedge \omega) - \langle v, \cdot \rangle \wedge (\iota_v \omega - \langle v, \cdot \rangle \wedge \omega) \\ &= -\iota_v(\langle v, \cdot \rangle \wedge \omega) + \langle v, \cdot \rangle \wedge \iota_v \omega \\ &= -|v|^2 \omega \end{aligned}$$

.♣

Remark 1.3. As $\mathbb{R}-$linear vector spaces, $\bigwedge V = \bigoplus_{k \geq 0} \bigwedge^k V$ is isomorphic to $C\ell(V)$ by expanding the following map:

$$\sigma : \bigwedge^k V \longrightarrow C\ell(V)$$

with $\sigma(\lambda \cdot v_1 \wedge \cdots \wedge v_k) = \lambda \cdot v_1 \cdot \cdots \cdot v_k$. However, this isomorphism is *not* an algebra isomorphism.

Moreover, if W is a Clifford module, then the Clifford action on W can also be extended to a *linear* $\bigwedge V$-action. ♣

Recall that from representation theory, a representation $\rho : C\ell(V) \longrightarrow \mathrm{End}_{\mathbb{C}}(W)$ is called *irreducible*, if W has no non-trivial $\rho-$invariant subspaces. Correspondingly, the irreducible representation space W is also called an *irreducible Clifford module*. It is called reducible if it is not irreducible. It is called *completely reducible*, if W can be decomposed into direct sums of 1-dimensional irreducible Clifford modules.

Here are several examples of irreducible Clifford modules.

Example 1.3. (1) The complex vector space $\$ = \mathbb{H}$ is an irreducible $C\ell(\mathbb{R}^3)-$module.

To see this, as was stated in Example 1.2 (1), the $\mathbb{R}^3 \cong \mathrm{Im}\mathbb{H}$ action on $\$$ is given by $h \mapsto qh$, where $q = q_1 i + q_2 j + q_3 k \in \mathbb{R}^3$, $h \in \$$. By the matrix presentation of quaternion (cf. Appendix 1.A equation (14)), we can write this Clifford multiplication as:

$$\begin{aligned} \varphi : &\mathbb{R}^3 \cong \mathrm{Im}\mathbb{H} \longrightarrow \mathrm{End}(\$) \\ &q \mapsto \begin{pmatrix} \sqrt{-1}q_3 & -q_1 + \sqrt{-1}q_2 \\ q_1 + \sqrt{-1}q_2 & -\sqrt{-1}q_3 \end{pmatrix} \end{aligned} \tag{6}$$

where the last matrix has two distinct eigenvalues $\pm\sqrt{-1}|q|$ for each q. Then by a theorem in representation theory, *a 2-dimensional representation is irreducible if and only if there are no common eigenvalues*, which ends the proof.

(2) The complex vector space $\$_+ \oplus \$_-$ is an irreducible $C\ell(\mathbb{R}^4)-$module.

In fact, as was shown in (4), the Clifford multiplication can be viewed as:

$$\phi : \mathbb{R}^4 \cong \mathbb{H} \longrightarrow \mathrm{End}(\$_+ \oplus \$_-),\ h \mapsto \begin{pmatrix} 0 & -\bar{h} \\ h & 0 \end{pmatrix} \tag{7}$$

where $h, -\bar{h}$ are presented by 2×2 complex matrices (cf. Appendix 1.A), we can compute the last matrix also has two distinct eigenvalues $\pm\sqrt{-1}|h|$ for each $h \in \mathbb{R}^4 \cong \mathbb{H}$, and each eigenvalue has a eigen-subspace of dimension 2, namely $E_h^{\pm}$. Hence $\$_+ \oplus \$_-$ is reducible if and only if $E_h^{\pm}$ are common eigen-subspaces for all $h \in \mathbb{H}$, which if and only if $\pm\sqrt{-1}|h|$ are common eigenvalues for all $h \in \mathbb{H}$, which is not the case. Therefore $\$_+ \oplus \$_-$ is an irreducible $C\ell(\mathbb{R}^4)-$module. ♣

Each representation of $C\ell(V)$ can be extended to a representation of its complexification $C\ell(V) \otimes_{\mathbb{R}} \mathbb{C}$.

Example 1.4. (1) The representation of $C\ell(\mathbb{R}^3)\otimes_{\mathbb{R}}\mathbb{C}$ on $\$$ can be obtained by expanding φ to

$$\varphi_{\mathbb{C}} : \mathrm{Im}\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathrm{End}(\$)$$

just by replacing each $q_i \in \mathbb{R}$ appeared in (6) to some complex numbers $q_i \in \mathbb{C}$.

Note that it is clearly that $\varphi_{\mathbb{C}}$ is faithful and the image are exactly the traceless endomorphisms of $\$$, denoted by $\mathrm{End}_0(\$)$, hence we obtained an isomorphism of $\mathbb{C}-$linear spaces

$$\mathrm{Im}\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{End}_0(\$) \tag{8}$$

(2) Similarly, one can also expand ϕ to $\phi_{\mathbb{C}} : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathrm{End}(\$_+ \oplus \$_-)$ to obtain a representation of $C\ell(\mathbb{R}^4)\otimes_{\mathbb{R}}\mathbb{C}$, by replacing h appeared in (7) to some 2×2 matrix. It also obviously that this representation of $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}$ is faithful as well, and the image can be identified with the $\mathbb{C}-$linear space $\mathrm{Hom}(\$_+; \$_-)$, hence we obtained

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Hom}(\$_+; \$_-) \tag{9}$$

. ♣

Luckily, the irreducible Clifford representations are not too much, and they can be classified completely:

Theorem 1.1. *Let V be a Euclidean space over $\mathbb{R}$.*

- *If $\dim V = 0 \mod 2$, then there is an unique irreducible representation of $C\ell(V)$; If $\dim V = 1 \mod 2$, then there are only 2 irreducible representations of $C\ell(V)$. The irreducible $C\ell(V)-$module has complex dimension $2^{\left[\frac{\dim V}{2}\right]}$, denoted by $\$_V$.*
- *Every irreducible representation of $C\ell(V)$ extends to a representation of $C\ell(V)\otimes_{\mathbb{R}}\mathbb{C}$ which is again irreducible.*

Table 1 is a list of the numbers of irreducible representations n_k and the dimensions of irreducible Clifford modules for each $C\ell(\mathbb{R}^k)$ $(k \leq 8)$.

We refer to [6, §1.5] for a detailed proof of theorem 1.1.

Table 1: Classification List of Irreducible Representations

k	$C\ell(\mathbb{R}^k)$	$C\ell(\mathbb{R}^k) \otimes_{\mathbb{R}} \mathbb{C}$	n_k	$\$_k$
1	$\mathbb{C}$	$\mathbb{C} \oplus \mathbb{C}$	2	$\mathbb{C}$
2	$\mathbb{H}$	$\mathrm{End}(\mathbb{C}^2)$	1	$\mathbb{C}^2$
3	$\mathbb{H} \oplus \mathbb{H}$	$\mathrm{End}(\mathbb{C}^2) \oplus \mathrm{End}(\mathbb{C}^2)$	2	$\$$
4	$\mathrm{End}(\mathbb{H}^2)$	$\mathrm{End}(\mathbb{C}^4)$	1	$\$_+ \oplus \$_-$
5	$\mathrm{End}(\mathbb{C}^4)$	$\mathrm{End}(\mathbb{C}^4) \oplus \mathrm{End}(\mathbb{C}^4)$	2	$\mathbb{C}^4$
6	$\mathrm{End}(\mathbb{R}^8)$	$\mathrm{End}(\mathbb{C}^8)$	1	$\mathbb{C}^8$
7	$\mathrm{End}(\mathbb{R}^8) \oplus \mathrm{End}(\mathbb{R}^8)$	$\mathrm{End}(\mathbb{C}^8) \oplus \mathrm{End}(\mathbb{C}^8)$	2	$\mathbb{C}^8$
8	$\mathrm{End}\,(\mathbb{R}^{16})$	$\mathrm{End}\,(\mathbb{C}^{16})$	1	$\mathbb{C}^{16}$

1.2 Spin Groups and Spin Representations

One important application of Clifford algebras is to construct rotations in $\mathbb{R}^n$.

Definition 1.3 (Spin group)**.** The *Spin group* $\mathrm{Spin}(V) \subset C\ell(V)$ is a group generated by the elements with the form $v_1 \cdot \cdots \cdot v_{2k}$, where $v_1, ..., v_{2k} \in V$ with $|v_i|^2 = 1$. ♣

Remark 1.4. Recall that by (1), each element $v \in V$ with $|v|^2 = 1$ has an inverse in $C\ell(V)$, namely $v^{-1} = -v$. ♣

Now, let $\sigma := v_1 \cdots v_{2k} \in \mathrm{Spin}(V)$, $w \in V$, let's compute what is $\sigma \cdot w \cdot \sigma^{-1}$. By applying (1) we have for any v_i:

$$\begin{aligned} v_i \cdot w \cdot v_i^{-1} &= -v_i \cdot w \cdot v_i = -v_i \cdot (-v_i \cdot w - 2\langle v_i, w\rangle) \\ &= -(w - 2\langle v_i, w\rangle v_i) \\ &= -R_{v_i}(w) \end{aligned}$$

where $R_{v_i}(w)$ means reflection of w along $v_i^{\perp}$. Therefore we have

$$\begin{aligned} \sigma \cdot w \cdot \sigma^{-1} &= (-1)^{2k} v_1 \cdots v_{2k} \cdot w \cdot v_1 \cdots v_{2k} \\ &= R_{v_{2k}} \circ \cdots \circ R_{v_1}(w) \end{aligned}$$

Recall that *Cartan-Dioudonneé theorem* asserts that every rotation in $\mathrm{SO}(V)$ can be decomposed as the even times compositions of reflections. Thus we obtained:

Lemma 1.2. *The group (real) representation*

$$\phi : \mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V),\ \sigma \mapsto \left(w \mapsto \sigma \cdot w \cdot \sigma^{-1}\right)$$

is an epimorphism. Moreover we have $\ker\phi = \mathbb{Z}_2 = \{\pm 1\}$, *hence*

$$\mathrm{Spin}(V)/\mathbb{Z}_2 \cong \mathrm{SO}(V)$$

which means $\mathrm{Spin}(V)$ *is a double cover of* $\mathrm{SO}(V)$.

Remark 1.5.

- Notice that $\mathbb{Z}_2 = \{\pm 1\} \subset \mathrm{Spin}(n)$ as the center, and ϕ maps $\mathbb{Z}_2$ to the identity matrix.
- For $\dim V \geq 3$, $\pi_1(\mathrm{SO}(V)) \cong \mathbb{Z}_2$, hence by double covering, $\pi_1(\mathrm{Spin}(V))$ is an index-2 subgroup of $\mathbb{Z}_2$, which is trivial, hence $\mathrm{Spin}(V)$ is simply connected and in fact an universal cover of $\mathrm{SO}(V)$.

Example 1.5.

(1) $\mathrm{Spin}(1) \cong \mathbb{Z}_2$. Since the unimodular vectors in $\mathbb{R}$ are just ± 1, they generate themselves in $C\ell(\mathbb{R}) \cong \mathbb{C}$.

(2) $\mathrm{Spin}(2) \cong S^1$. Since the unimodular vectors in $\mathbb{R}^2$ are the unit circle S^1, which is embedded in the $(i, j)-$plane in $\mathbb{H}$ (see Example 1.1 (2)), hence it generates itself.

(3) $\mathrm{Spin}(3) \cong \mathrm{U}(1, \mathbb{H}) \cong \mathrm{Sp}(1)$. Since $\mathbb{R}^3 \cong \mathrm{Im}\mathbb{H}$ is diagonally embedded into $\mathbb{H} \oplus \mathbb{H} \cong C\ell(\mathbb{R}^3)$ via $q \mapsto (-q, q)$ (cf. Example 1.1 (3)). Hence $\mathrm{Spin}(3)$ is generated by even-times product of the elements with the form $(-q, q)$, where $|q|^2 = 1$. Which is equivalent to be generated by $q \in \mathrm{Im}\mathbb{H}$ in $\mathbb{H}$ (since product even-times kills the negative sign), which is $\mathrm{U}(1, \mathbb{H})$. Topologically it is an $S^3 \cong \mathrm{SU}(2)$, indeed a double cover of $\mathbb{RP}^3 \cong \mathrm{SO}(3)$.

(4) $\mathrm{Spin}(4) \cong \mathrm{Spin}(3) \times \mathrm{Spin}(3)$. Because $\mathbb{R}^4 \cong \mathbb{H}$ is embedded in $\mathrm{End}(\mathbb{H}^2) \cong C\ell(\mathbb{R}^4)$ via $h \mapsto \begin{pmatrix} 0 & -\bar{h} \\ h & 0 \end{pmatrix}$ (cf. Example 1.1 (4)), and even-times products of the elements of the anti-diagonal matrices will be diagonal, thus $\mathrm{Spin}(4)$ is actually generated by unit quaternion in $\mathbb{H} \oplus \mathbb{H}$, which is $\mathrm{Spin}(3) \times \mathrm{Spin}(3)$. ♣

Remark 1.6. It will be convenient to write

$$\mathrm{Spin}(4) \cong \mathrm{Spin}(3) \times \mathrm{Spin}(3) \cong \mathrm{SU}_+(2) \times \mathrm{SU}_-(2)$$

since it will help us to write down the Spin representations simply by matrix multiplication. ♣

As a Lie group, the Lie algebra $\mathfrak{spin}(n)$ is isomorphic to $\mathfrak{so}(n)$ according to lemma 1.2. Then it is natural to study the exponential map $\exp : \mathfrak{spin}(n) \longrightarrow \mathrm{Spin}(n)$. Since $\mathrm{Spin}(n) \subset C\ell(\mathbb{R}^n)$, we hope the value $\exp X$ can be presented by elements in Clifford algebra, so that the rotation action of $\exp X$ on $\mathbb{R}^n$ can be presented by multiplication in $C\ell(\mathbb{R}^n)$.

Note that $\mathfrak{so}(n)$ is identified with $\bigwedge^2 \mathbb{R}^n$, the basis can be write as $e_i \wedge e_j$, where e_i are orthonormal basis in $\mathbb{R}^n$. It stands for a matrix $E_{ij} - E_{ji}$.

Theorem 1.2. *The exponential map is computed as*

$$\begin{aligned} \exp : \mathfrak{spin}(n) &\longrightarrow \mathrm{Spin}(n) \subset \mathrm{C}\ell(\mathbb{R}^n) \\ \exp(\theta(e_i \wedge e_j)) &= \cos\frac{\theta}{2} + \sin\frac{\theta}{2} e_i \cdot e_j \end{aligned} \tag{10}$$

proof. Notice that in $\mathrm{SO}(n)$, $\exp(\theta(e_i \wedge e_j))$ represents for the rotation in (i,j)-plane, rotating along $e_i \to e_j$ by θ-angle. Then it suffices to notice that the action

$$w \mapsto \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2} e_i \cdot e_j\right) \cdot w \cdot \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2} e_i \cdot e_j\right)^{-1}$$

precisely means the rotation in (i,j)-plane, rotating along $e_i \to e_j$ by θ-angle. That will prove the claim. ♣

Another important fact about spin groups is their representations.

Theorem 1.3. *There is a unique complex representation of* $\mathrm{Spin}(V)$, *denoted by*

$$\rho : \mathrm{Spin}(V) \longrightarrow \mathrm{GL}(\$_V)$$

distinguished by the property: ρ can be extended to an irreducible representation of $\mathrm{C}\ell(V)$ *(cf. Theorem 1.1).*

*Such a representation is called the **Spin representation** of* $\mathrm{Spin}(V)$*, the elements in* $\$_V$ *are called **spinors**.*

We refer to [2, §2] [6, §1] for a detailed proof. We shall just construct some examples.

Remark 1.7. The spin representation *doesn't* imply that it is either irreducible or the unique $\mathrm{Spin}(n)$-representation. In fact, when $\dim V = 2k$ is even, the spin representation is not irreducible, since $\$_{2k}$ will split into two sub-representation, namely $\$_{2k_+} \oplus \$_{2k_-}$.

Example 1.6. (i) The spin representation of $\mathrm{Spin}(3) \cong \mathrm{U}(1,\mathbb{H}) \cong \mathrm{SU}(2)$ is simply

$$\rho : \mathrm{Spin}(3) \cong \mathrm{SU}(2) \longrightarrow \mathrm{GL}(\$),\ A \mapsto A$$

which is an irreducible representation.

(ii) The spin representation of $\mathrm{Spin}(4) \cong \mathrm{SU}_+(2) \times \mathrm{SU}_-(2)$ (cf. remark 1.6) is:

$$\rho : \mathrm{Spin}(4) \cong \mathrm{SU}_+(2) \times \mathrm{SU}_-(2) \longrightarrow \mathrm{GL}(\$_+ \oplus \$_-)$$

by

$$(h_+, h_-) \mapsto (h_+, h_-) \begin{pmatrix} A_+ & \\ & A_- \end{pmatrix}$$

which is *reducible* with two irreducible subspaces $\$_\pm$ respectively, the restricted irreducible representation of on $\$_\pm$ will be denoted by $\rho_\pm$, hence $\rho = \rho_+ \oplus \rho_-$. ♣

Remark 1.8.
- Note that $\rho(\pm 1) = \pm \mathrm{Id}_{\$_V}$ by the property ρ extends to an irreducible Clifford representation.
- We can equip with $\$_V$ an Hermitian inner product so that it is $\mathrm{Spin}(V)$-invariant, hence the spin representation ρ is in fact taking values in $\mathrm{U}(\$_V)$.

1.3 Spin^c Groups

It is natural to define a subgroup in the complexification algebra $C\ell(V) \otimes_{\mathbb{R}} \mathbb{C}$.

Definition 1.4. The group $\mathrm{Spin}^c(n) \subset C\ell(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$ is the subgroup generated by $\mathrm{Spin}(n)$ and the unit complex numbers $\mathrm{U}(1)$.

Remark 1.9. (a) Apparently,

$$\begin{aligned} \mathrm{Spin}^c(n) &= \{\lambda \otimes \sigma | \lambda \in \mathrm{U}(1), \sigma \in \mathrm{Spin}(n)\} \\ &:= \mathrm{U}(1) \cdot \mathrm{Spin}(n) \cong \mathrm{Spin}(n) \times_{\mathbb{Z}_2} \mathrm{U}(1) \end{aligned} \tag{11}$$

Since we can define a map

$$\phi : \mathrm{Spin}(n) \times \mathrm{U}(1) \longrightarrow \mathrm{Spin}^c(n),\ (\sigma, \lambda) \mapsto \lambda \otimes \sigma$$

which is an epimorphism and $\ker \phi$ are of the form $(\lambda \cdot 1, \lambda^{-1})$, $\lambda \in \mathbb{R}$, hence it could only be ± 1.

(b) As a notation convention, an element in $\mathrm{Spin}^c(n)$ will be denoted by $\lambda \otimes \sigma$, and $\mathrm{U}(1)$ is embedded in $\mathrm{Spin}^c(n)$ as the center.

(c) Like the real case (cf. lemma 1.2), $\mathrm{Spin}^c(n)$ also admits a well-defined real representation:

$$\phi^c : \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n)$$

by

$$\lambda \otimes \sigma \mapsto \left(v \mapsto \sigma \cdot v \cdot \sigma^{-1}\right)$$

since $\pm\sigma$ yields the same representation.

Moreover, ϕ^c is full and $\ker \phi^c \cong \mathrm{U}(1)$, hence we have a short exact sequence of Lie groups

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{Spin}^c(n) \xrightarrow{\phi^c} \mathrm{SO}(n) \longrightarrow 1$$

(d) By (11), the Lie algebra of $\mathrm{Spin}^c(n)$ is simply

$$\mathfrak{spin}(n) \oplus i\mathbb{R} \cong \mathfrak{so}(n) \oplus i\mathbb{R}$$

hence by (10), we can compute the exponential map by

$$\exp(\theta(e_i \wedge e_j, it)) = \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2} e_i \cdot e_j\right) \otimes e^{it\theta}$$

Example 1.7. (i) For $\dim V = 3$, we have

$$\mathrm{Spin}^c(3) \cong \mathrm{U}(1) \cdot \mathrm{SU}(2) \cong \mathrm{U}(2)$$

Therefore, we see that $\mathrm{Spin}^c(n)$ is *not* the complexification of $\mathrm{Spin}(n)$, since Spin^c group may not even be complex Lie group.

(ii) For $\dim V = 4$, we have

$$\begin{aligned}\operatorname{Spin}^c(4) &\cong \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathbb{Z}_2}\\ &\cong \left\{\begin{pmatrix}\lambda A_+ & O\\ O & \lambda A_-\end{pmatrix}\middle| \lambda\in\mathrm{U}(1), A_\pm\in\mathrm{SU}_\pm(2)\right\}\\ &= \left\{\begin{pmatrix}A_+ & O\\ O & A_-\end{pmatrix}\in\mathrm{U}_+(2)\times\mathrm{U}_-(2)\middle| \det A_+ = \det A_-\right\}\end{aligned}$$

Similar to the spin group (cf. Theorem 1.3), Spin^c groups also have a distinguished spin representation.

Theorem 1.4. *Let $\rho : \operatorname{Spin}(V) \longrightarrow \mathrm{GL}(\$_V)$ be a spin representation, then ρ extends* ***uniquely*** *to the representation*

$$\rho^c : \operatorname{Spin}^c(V) \longrightarrow \mathrm{GL}(\$_V)$$

such a distinguished representation will still be called the spin representation of $\operatorname{Spin}^c(V)$.

Hence by theorem 1.1, we have $\dim_{\mathbb{C}} \$_V = 2^{\left[\frac{\dim V}{2}\right]}$. A proof can be found in [2, §2.6].

Remark 1.10. (a) Like the $\operatorname{Spin}(n)$ case, $\$_V$ also admits a $\operatorname{Spin}^c(n)-$invariant Hermitian inner product, see remark 1.8, hence ρ^c is actually taking values in $\mathrm{U}(\$_V)$.

(b) There is a well-defined surjection

$$\begin{aligned}\rho_{\det} : \operatorname{Spin}^c(n) &\longrightarrow \mathrm{U}(1)/\mathbb{Z}_2 \xrightarrow{\cong} \mathrm{U}(1)\\ \lambda\otimes\sigma &\mapsto [\lambda\det\rho(\sigma)] \mapsto (\lambda\det\rho(\sigma))^2\end{aligned}$$

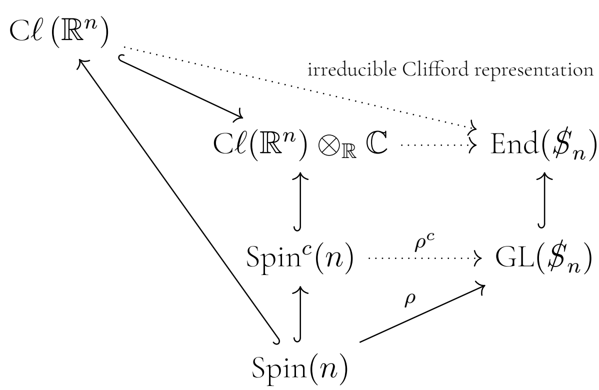
whose kernel is precisely $\operatorname{Spin}(n)$. It is well-defined, since for different presentation $-\lambda\otimes(-\sigma)$,we have

$$\rho_{\det}((-\lambda)\otimes(-\sigma)) = (-\lambda\det(-\rho(\sigma)))^2 = (\lambda\det\rho(\sigma))^2$$

hence we have a short exact sequence of Lie groups:

$$1 \longrightarrow \operatorname{Spin}(n) \hookrightarrow \operatorname{Spin}^c(n) \overset{\rho_{\det}}{\twoheadrightarrow} \mathrm{U}(1) \longrightarrow 1$$

(c) Note that for $\lambda\otimes 1 \in \mathrm{U}(1) \subset \operatorname{Spin}^c(n)$, we have $\rho^c(\lambda\otimes 1) = \lambda\cdot\mathrm{Id}_{\$_n}$ due to the fact that ρ^c also extends to an irreducible $C\ell(\mathbb{R}^n)\otimes_{\mathbb{R}}\mathbb{C}$ representation. See following diagram.



Here are examples for spin representations for $n = 3, 4$, and the computations of $\rho_{\det}$.

Example 1.8. (1) The spin representation of $\mathrm{Spin}^c(3) \cong \mathrm{U}(2)$ is simply trivial, since it is taking value in $\mathrm{U}(\$) \cong \mathrm{U}(2)$, ρ^c is just the identity.

As for $\rho_{\det}$, write an element in $\mathrm{Spin}^c(3)$ as λA, where $A \in \mathrm{SU}(2)$, hence by remark 1.10 (b), we have

$$\rho_{\det}(\lambda A) = \lambda^2 \det A = \lambda^2$$

hence $\rho_{\det}$ is just taking the determinant of matrices in $\mathrm{U}(2) \cong \mathrm{Spin}^c(3)$.

(2) The spin representation of $\mathrm{Spin}^c(4)$ is again trivial

$$(w_+, w_-) \in \$_+ \oplus \$_- \mapsto (w_+, w_-) \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix}$$

which is not irreducible. It is also convenient to define two irreducible sub-representations $\rho^c_\pm$ on $\$_\pm$ (cf. Example 1.6 (ii)):

$$\rho^c_\pm \begin{pmatrix} \lambda A_+ & O \\ O & \lambda A_- \end{pmatrix} = \lambda A_\pm \in \mathrm{U}(\$_\pm)$$

As for $\rho_{\det}$, we compute by definition that

$$\rho_{\det} \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} = \lambda^2 (\det A_+)(\det A_-) = \det \lambda^2$$

thus if we write elements in $\mathrm{Spin}^c(4)$ as $(A_+, A_-) = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}$ where $A_\pm \in \mathrm{U}(2)$ and $\det A_+ = \det A_-$, then we have [4]

$$\rho_{\det}(A_+, A_-) = \det A_+ = \det A_-$$

Appendix 1.A The Quaternion Algebra $\mathbb{H}$

Definition 1.5 (Quaternion)**.** The set of quaternion numbers $\mathbb{H}$ is an unital $\mathbb{R}-$algebra generated by $1, i, j, k$, in which i, j, k are called the *imaginary unit*, subordinate to the relation

$$j^2 = j^2 = k^2 = ijk = -1$$

A quaternion number is an element $q \in \mathbb{H}$, which can be presented by

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

where $q_i \in \mathbb{R}$. ♣

[4] In general, $\rho_{\det}$ can have nothing to do with the determinant. For example

$$\mathrm{Spin}^c(6) = \mathrm{SU}(4) \times_{\mathbb{Z}_2} \mathrm{U}(1) = \mathrm{U}(4)$$

the spin representation is again the trivial one, so we have $\rho_{\det}(\lambda A) = \lambda^2$, but $\det \lambda A = \lambda^4$.

Remark 1.11. (1) It is convenient to embed $\mathbb{R}^3$ into $\mathbb{H}$ by identifying $\mathbb{R}^3 \cong \mathrm{Im}\mathbb{H}$. Under this setting, it is convenient to denote a quaternion by $q = q_0 + \boldsymbol{q}$, where q_0 is called the *scalar part* or *real part*, $\boldsymbol{q}$ is called the *vector part* or *imaginary part*.

Notice that for two pure imaginary quaternions $\boldsymbol{q}, \boldsymbol{h} \in \mathbb{R}^3 \cong \mathrm{Im}\mathbb{H}$, we can compute their multiplication by

$$\boldsymbol{q}\boldsymbol{h} = -\boldsymbol{q} \cdot \boldsymbol{h} + \boldsymbol{q} \times \boldsymbol{h}$$

where $\cdot$ and $\times$ are inner and exterior product in $\mathbb{R}^3$ respectively. Therefore, the multiplication of two quaternions $q, h \in \mathbb{H}$ can be write as

$$qh = (q_0 h_0 - \boldsymbol{q} \cdot \boldsymbol{h}) + (q_0 \boldsymbol{h} + h_0 \boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{h}) \tag{12}$$

(2) Like complex numbers, we can also define the *conjugate* of a quaternion $q = q_0 + \boldsymbol{q}$ by $\bar{q} := q_0 - \boldsymbol{q}$, and the square-module of q is defined by $|q|^2 := q\bar{q}$, which is indeed a positive real number by equation (12).

(3) Notice that $\mathbb{H}$ is a *divisible algebra*[5]. That is for any non-zero quaternion $q \neq 0$, there exists an inverse q^{-1} such that $qq^{-1} = q^{-1}q = 1$. Here the inverse is simply $q/|q|^2$, in particular, if $q \in \mathrm{U}(1, \mathbb{H})$ which is a unit quaternion, then $\bar{q} = q^{-1}$. ♣

Quaternion can be used to present rotations in $\mathbb{R}^3$.

Proposition 1.1. *(i) Every rotation in* $\mathrm{SO}(3)$ *can be presented by* $h \mapsto qh\bar{q}$ *where* $q \in \mathrm{U}(1, \mathbb{H})$.

(ii) Every rotation in $\mathrm{SO}(4)$ *can be presented by* $h \mapsto q_+ h q_-$ *where* $q_\pm \in \mathrm{U}(1, \mathbb{H})$.

Like complex numbers, a quaternion also admits a matrix presentation[6]. In fact, denoted by

$$i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; j = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \; k = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \tag{13}$$

then the quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$ can be write by

$$q = \begin{pmatrix} q_0 + \sqrt{-1}q_3 & -\overline{q_1 + \sqrt{-1}q_2} \\ q_1 + \sqrt{-1}q_2 & \overline{q_0 + \sqrt{-1}q_3} \end{pmatrix} \in \mathrm{End}(\mathbb{C}^2) \tag{14}$$

hence the algebra $\mathbb{H}$ is embedded as an $\mathbb{R}-$subalgebra in $\mathrm{End}(\mathbb{C}^2)$.

Moreover, one can find that

$$\det q = \sum_{i=0}^{3} q_i^2 = |q|^2$$

The matrix presentation can help us to compute the complexification $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, which was claimed in example 1.1 part (2).

[5]In fact, the only divisible associative algebra over $\mathbb{R}$ are of dimension 1,2 and 4. The only divisible non-associative algebra over $\mathbb{R}$ is the octonion $\mathbb{O}$, which is of dimension 8.

[6]Algebraically, when we say an algebra A admits a $k-$matrix presentation, that implies there is an irreducible $k-$representation of A into $\mathrm{End}_k(W)$, where W is a $k-$vector space.

Proposition 1.2 (Complexification)**.** *The complexification* $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}$ *is isomorphic to* $\mathrm{End}(\mathbb{C}\oplus\mathbb{C})$.

proof. Recall that $\mathrm{End}(\mathbb{C}\oplus\mathbb{C})$ as an $\mathbb{R}-$algebra is generated by $1, i, j, k$ which was defined in (13) together with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, as a $\mathbb{C}-$algebra, it is generated by $1, i$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Now, as an $\mathbb{R}-$subalgebra, $\mathbb{H}$ is generated by $1, i, j, k$, by complexification, it allows us to multiple with $\sqrt{-1}$, that yields the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, hence $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}\cong\mathrm{End}(\mathbb{C}\oplus\mathbb{C})$ as an isomorphism between $\mathbb{C}-$algebras. ♣

Intuitively, an element of the complexification $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}$ can be simply understood as replacing each $q_i\in\mathbb{R}$ in (14) to some complex numbers $q_i\in\mathbb{C}$.

Remark 1.12. In particular, if we regard $\mathbb{H}$ as a complex *vector space* $\not S$ (cf. example 1.2 (1)), we have

$$\mathrm{End}(\not S) = \mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}$$

. ♣

2 Spin Geometry and Dirac Operators

In what follows, we assume (M, g) is an $n-$dimensional *compact oriented* [7] Riemannian manifold. Let P be the frame bundle of M, which is a principal $\mathrm{SO}(n)-$bundle on M under our settings. Let

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{SO}(n)$$

be the transition functions of P, where $\{U_\alpha\}$ should be chose to be a *good cover* of M.

We will globalize all constructions in § 1 to the geometric objects on M.

2.1 Spin Structures and Spinor Bundles

Definition 2.1 (Spin Structure)**.** A spin structure on (M, g) is a choice of $\mathrm{Spin}(n)-$lifting of the transition functions:

$$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Spin}(n)$$

in the sense that $\phi \circ \tilde{g}_{\alpha\beta} = g_{\alpha\beta}$ and such that $\{\tilde{g}_{\alpha\beta}\}$ can define a principal $\mathrm{Spin}(n)-$bundle $\tilde{P}$. Where ϕ is the standard real representation of $\mathrm{Spin}(n)$ defined in lemma 1.2. ♣

We can see that if (M, g) admits a spin structure $\tilde{P}$, then $\tilde{P} \longrightarrow P$ is a double cover.

Now, if (M, g) is a spin manifold with a spin structure $\tilde{P}$, then by theorem 1.3, the structure group $\mathrm{Spin}(n)$ has a distinguished spin representation ρ on $\$_n$, so we can define an adjoint bundle of $\tilde{P}$.

Definition 2.2 (Spinor Bundle)**.** If (M, g) admits a spin structure $\tilde{P}$, the adjoint bundle

$$\$_M := \tilde{P} \times_{\rho, \mathrm{Spin}(n)} \$_n$$

is called the *spinor bundle* of M with respects to the chosen spin structure. A section $\psi \in \Gamma(\$_M)$ is called a *spinor field* or simply a *spinor*. ♣

Remark 2.1.

- Recall that by remark 1.8, the fiber $\$_n$ is a complex vector space endowed with a $\mathrm{Spin}(n)-$invariant Hermitian metric, hence $\$_M$ is in fact an Hermitian vector bundle with rank $2^{\left[\frac{\dim M}{2}\right]}$ (cf. Theorem 1.1), where the Hermitian metric can be defined point-wisely. In particular it has $\mathrm{U}\left(2^{\left[\frac{\dim M}{2}\right]}\right)$ as the structure group.
- The transition function of $\$_M$ is simply

$$\rho \circ \tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{U}(\$_n)$$

and the transition of $\det \$_M$ is simply

$$\det \rho \left(\tilde{g}_{\alpha\beta}\right) : U_\alpha \cap U_\beta \longrightarrow \mathrm{U}(\$_M) \xrightarrow{\det} \mathrm{U}(1)$$

[7] A manifold M is oriented is equivalent to say its first Stiefel-Whitney class vanishes $w_1(M) = 0$.

- We call two spin structures $\tilde{P}$, $\tilde{P}'$ are equivalent if they are isomorphic as principle bundles, hence it is equivalent to say that they have the same spinor bundles.

Example 2.1. (1) When $\dim M = 3$, as was shown in Example 1.6 (i), the spinor bundle of M is a rank 2 Hermitian vector bundle whose fiber is $\not{S}$, the spinor bundle will still be denoted by $\not{S}$.

Since $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$, we can see that the determinant line bundle $\det \not{S} = \bigwedge^2 \not{S}$ has the transition function (cf. Example 1.6 (i)):

$$\det(\rho(\tilde{g}_{\alpha\beta})) = \det \tilde{g}_{\alpha\beta} = 1$$

thus $\det \not{S}$ of a 3-manifold is the trivial line bundle $M \times \mathbb{C}$, and hence $\not{S}$ has the structure group $\mathrm{SU}(2)$.

Due to the simply connectedness of $\mathrm{SU}(2) \cong S^3$ and $\dim M = 3$, I conclude that $\not{S}$ is trivial by an obstruction theoretical argument.

Recall that by (8) in Example 1.4, we have

$$TM \otimes \mathbb{C} \cong \mathrm{End}_0(\not{S}) \cong \bigwedge^2 (TM \otimes \mathbb{C})$$

(2) If $\dim M = 4$, as was shown in Example 1.6 (ii), the spinor bundle of M is a splitting rank 4 Hermitian vector bundle

$$\not{S} = \not{S}_+ \oplus \not{S}_-$$

where $\not{S}_\pm$ are rank 2 Hermitian bundles, induced by $\rho_\pm$ respectively (cf. example 1.6 (ii)). Sections in $\Gamma(\not{S}_\pm)$ are called spinor fields with *positive* or *negative chirality* respectively.

Similarly, by Example 1.6 (ii), we see that for a 4-manifold, the determinant line bundle of its spinor bundle is trivial [8]:

$$\bigwedge^4 (\not{S}_+ \oplus \not{S}_-) = \det \not{S}_+ = \det \not{S}_- = M \times \mathbb{C}$$

and by (9), we have

$$TM \otimes \mathbb{C} \cong \mathrm{Hom}(\not{S}_+; \not{S}_-)$$

However, not every compact oriented Riemannian manifold admits a spin structure, the obstruction was described by the second Stiefel-Whitney class $w_2(M)$.

[8] We will see later (cf. Example 2.3 (1)) that for any spin manifold M, the first Chern class of $\not{S}_M$ associated to a spin structure satisfies

$$0 = c_1(\not{S}_M) \mod 2$$

Theorem 2.1.
- *An oriented Riemannian manifold (M,g) admits a spin structure if and only if $w_2(M)=0$.*
- *If (M,g) admits a spin structure, then all possible spin structures are classified by cohomology classes in $H^1(M;\mathbb{Z}_2)$, that is they were classified by real line bundles over M.*

Before proving the theorem, let's first recall some notions in characteristic classes, I'd like to refer to the excellent books [7–9] for more about this interesting topic.

Recall that all real vector bundles over (M,g) are 1-1 correspondent to the principal $\mathrm{O}(n)-$bundles over M, denoted by $\mathrm{Prin}_{\mathrm{O}(n)}(M)$, which by homotopy theory, are 1-1 correspondent to

$$\mathrm{Prin}_{\mathrm{O}(n)}(M) \overset{1-1}{\longleftrightarrow} [M; B\mathrm{O}(n)]$$

In here, $B\mathrm{O}(n) \cong B\mathrm{GL}(n,\mathbb{R}) \cong \mathrm{Gr}_n(\mathbb{R}^\infty)$ is the classifying space of $\mathrm{O}(n)$, and we have the cohomology ring

$$H^*(\mathrm{Gr}_n(\mathbb{R}^\infty);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1,...,w_n]$$

where those $w_k \in H^k(\mathrm{Gr}_n(\mathbb{R}^\infty);\mathbb{Z}_2) \cong \mathbb{Z}_2$ are the generators.

Now, for a real vector bundle E, let $f_E \in [M;\mathrm{Gr}_n(\mathbb{R}^\infty)]$ be the *unique* map distinguished by the property that E is the pull-back of the universal bundle $E\mathrm{O}(n)$ along f_E, denoted by f_E^* the pull-back homomorphism between

$$f_E^* : H^k(\mathrm{Gr}_n(\mathbb{R}^\infty);\mathbb{Z}_2) \longrightarrow H^k(M;\mathbb{Z}_2)$$

Definition 2.3 (Stiefel-Whitney Classes)**.** The $k-$th Stiefel-Whitney class $w_k(E)$ is defined to be the pull-back of $f_E^*(w_k) \in H^k(M;\mathbb{Z}_2)$. ♣

Proof of Theorem 2.1. Notice that by lemma 1.2, there is a short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \overset{\phi}{\longrightarrow} \mathrm{SO}(n) \longrightarrow 1$$

which induces a short exact sequences between non-Abelian sheaves:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathcal{S}pin(n) \overset{\phi}{\longrightarrow} \mathcal{SO}(n) \longrightarrow 1$$

where $\mathcal{S}pin(n), \mathcal{SO}(n)$ stands for the sheaves of smooth functions over (M,g) taking values in $\mathrm{Spin}(n)$ and $\mathrm{SO}(n)$ respectively. $\mathbb{Z}_2$ stands for the locally constant sheaf and ϕ is the double cover which was defined in lemma 1.2.

It induces a (not too) long exact sequence in cohomology[9]:

$$\cdots \to H^1(M;\mathbb{Z}_2) \longrightarrow H^1(M;\mathcal{S}pin(n)) \overset{\phi^*}{\longrightarrow} H^1(M;\mathcal{SO}(n)) \overset{w_2}{\longrightarrow} H^2(M;\mathbb{Z}_2) \quad (15)$$

[9] For short exact sequence of non-Abelian sheaves

$$1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 1$$

the induced long exact sequence will stop at

$$0 \longrightarrow H^0(X;\mathcal{F}) \longrightarrow \cdots \longrightarrow H^1(X;\mathcal{H})$$

which is to say the functor H^2 will lost its exactness (the definition of H^k is similar as the cocycles of vector bundles). But here we can extend it to $H^2(M;\mathbb{Z}_2)$ because $\mathbb{Z}_2$ is the usual Abelian sheaf.

We see that an $\mathrm{SO}(n)-$bundle $E = \{g_{\alpha\beta}\} \in H^1(M; \mathcal{SO}(n))$ admits a $\mathrm{Spin}(n)-$lifting, if and only if it has a pre-image under ϕ^*, which if and only if, due to the exactness of (15), $w_2(E) = 0$.

We claim that w_2 is actually taking the second Stiefel-Whitney class. To see this, we will generalize the construction in Definition 2.3 for $\mathrm{SO}(n)$-bundles.

For each E, Let $g_E \in [M; B\mathrm{SO}(n)]$ be the unique map such that $E = g_E^*(E\mathrm{SO}(n))$. We shall claim that

(i) The cohomology ring of $B\mathrm{SO}(n)$ is

$$H^*(B\mathrm{SO}(n); \mathbb{Z}_2) \cong \mathbb{Z}_2[\omega_2, ..., \omega_k]$$

where $\omega_k \in H^k(B\mathrm{SO}(n); \mathbb{Z}_2) \cong \mathbb{Z}_2$ are generators for $k \geq 2$, and

$$H^1(B\mathrm{SO}(n); \mathbb{Z}_2) = 0$$

(ii) The pull-back cohomology classes $g_E^*(\omega_k) \in H^k(M; \mathbb{Z}_2)$ coincide with $f_E^*(w_k)$, where $f_E \in [M; B\mathrm{O}(n)]$ is unique mapping such that E is the pull-back of the universal bundle $E\mathrm{O}(n)$ along f_E.

If claim (i) & (ii) can be proved, then the result follows directly by applying (15) to the classifying space $B\mathrm{SO}(n)$, and notice that g_E^* induces the following commutative diagram of exact sequences:

$$\begin{array}{ccccc}
H^1(M; \mathcal{S}pin(n)) & \xrightarrow{\phi^*} & H^1(M; \mathcal{SO}(n)) & \xrightarrow{w_2} & H^2(M; \mathbb{Z}_2) \\
{\scriptstyle g_E^*}\uparrow & & {\scriptstyle g_E^*}\uparrow & & {\scriptstyle g_E^*}\uparrow \\
H^1(B\mathrm{SO}(n); \mathcal{S}pin(n)) & \longrightarrow & H^1(B\mathrm{SO}(n); \mathcal{SO}(n)) & \xrightarrow{w_2} & H^2(B\mathrm{SO}(n); \mathbb{Z}_2) \\
 & & & & \| \\
 & & & & \mathbb{Z}_2
\end{array}$$

hence the w_2 in the first line is indeed taking second Stiefel-Whitney class of E, because the w_2 in the second line sends $E\mathrm{SO}(n)$ to the generator.

Proof of (i) can be done by induction, which can be found in [10, Theorem 1.3]. Claim (ii) is a consequence of (i), since the $\omega_k(E)$ defined by $g_E^*(\omega_k)$ should satisfy the axioms of Stiefel-Whitney classes, hence the result follows by the uniqueness.

For the classification of $\mathrm{Spin}(n)-$structures of $P = (g_{\alpha\beta})$, we first fix a spin structure $\tilde{g}_{\alpha\beta}$ as a frame point whose spinor bundle is denoted by $\$_M$. Let L be a real line bundle with the transition functions

$$r_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(1, \mathbb{R}) \cong \mathbb{Z}_2 = \{\pm 1\} \subset \mathrm{Spin}(n)$$

We can define a new Spin structure by twisted the original one by L, namely $r_{\alpha\beta}\tilde{g}_{\alpha\beta}$. It indeed defined a spin structure since it is clearly satisfying the cocycle condition and

$$\phi\left(r_{\alpha\beta}\tilde{g}_{\alpha\beta}\right) = \phi(r_{\alpha\beta})\phi\left(\tilde{g}_{\alpha\beta}\right) = \phi\left(\tilde{g}_{\alpha\beta}\right) = g_{\alpha\beta}$$

Let $\$'_M$ be the spinor bundle associated to the twisted spin structure, then we find it has transition functions (cf. Remark 1.8)

$$\rho\left(r_{\alpha\beta}\tilde{g}_{\alpha\beta}\right) = \rho(r_{\alpha\beta})\rho\left(\tilde{g}_{\alpha\beta}\right) = r_{\alpha\beta}\rho\left(\tilde{g}_{\alpha\beta}\right)$$

thus $\$'_M = \$_M \otimes L$ in particular, the novel twisted spin structure coincides with the original one if and only if their spinor bundles coincide, hence which if and only if L is trivial. Hence all spin structures are classified by the real line bundles, and in particular it is an affine space modeled on $H^1(M;\mathbb{Z}_2)$ after picking a frame point. ♣

Some other proofs can be found in [11, §7.5.6] [6, §2.1] or Qiaochu Yuan's answer.

Example 2.2.
- If (M, g) is a 3-manifold, then M admits a spin structure. Because every compact oriented 3-manifold has trivial tangent bundle, which can be proved by applying Wu's formula to compute $w_2(M) = 0$ (see [12] [7, Exercise 12.4][10] or here and here).
- $M = \mathbb{CP}^2$ has no spin structures, since

$$w_2(\mathbb{P}^2) = c_1\left(\mathbb{P}^2\right) \mod 2$$

and $c_1(\mathbb{P}^2) \in H^2(\mathbb{P}^2;\mathbb{Z}) \cong \mathbb{Z}$ is the generator, which is not zero modulo 2. ♣

2.2 Spin^c Structures

Definition 2.4. A Spin^c−structure on (M, g) is a lifting of the transition functions $g_{\alpha\beta}$ of P to

$$\tilde{g}^c_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Spin}^c(n)$$

in the sense that $\phi^c \circ \tilde{g}^c_{\alpha\beta} = g_{\alpha\beta}$, and $\tilde{g}^c_{\alpha\beta}$ satisfy the cocycles condition so that it defines a Spin$^c(n)$-bundle P^c. Here ϕ^c is the standard real presentation of Spin$^c(n)$ defined in remark 1.9 (c). ♣

Like the spin structures, if (M, g) admits a Spin^c structure, then by Theorem 1.4 we can also associate it with an adjoint Hermitian vector bundle.

Definition 2.5. Let P^c be a Spin^c−structure on (M, g), then the adjoint bundle

$$P^c \times_{\rho^c,\mathrm{Spin}^c(n)} \$_n$$

will still be called the *spinor bundle* of M with respect to the chosen Spin^c−structure, which will still be denoted by $\$_M$. Here ρ^c is the unique spin representation of Spin$^c(n)$ which extends the spin representation of Spin(n), see theorem 1.4. ♣

Different form the spin case, the determinant line bundle of a Spin^c structure is a bit subtle.

[10] An answer can be found at this Math Stack Exchange post.

Definition 2.6. The *determinant line bundle of the* Spin^c structure [11] is defined to be

$$\mathcal{L} := P^c \times_{\rho_{\det}} \mathbb{C}$$

where $\rho_{\det}$ was defined in remark 1.10 (b). ♣

The existence of a Spin^c structure is much easier than the spin structure, since the map ϕ^c is not a double cover, it has kernel $\mathrm{U}(1) \cong S^1$, hence P^c can be viewed as a circle bundle [12] over P.

Theorem 2.2. *(i)* *(M, g) admits a Spin^c−structure if and only if $w_2(M)$ is a modulo 2 reduction of some $c_M \in H^2(M;\mathbb{Z})$* [13].

More precisely, if (M, g) has a Spin^c structure, then the first Chern class $c_1(\mathcal{L}) \in H^2(M;\mathbb{Z})$ of the determinant line bundle of the associated Spin^c structure satisfies

$$w_2(M) = c_1(\mathcal{L}) \mod 2$$

Conversely, if $w_2(M)$ is a modulo 2 reduction of some $c_1(\mathcal{L}) \in H^2(M;\mathbb{Z}_2)$, then there exists a Spin^c structure on M such that its determinant bundle equals to $\mathcal{L}$.

(ii) *If (M, g) is a Spin^c manifold, then all possible Spin^c structures are classified by the cohomology classes in $H^2(M;\mathbb{Z})$, that is they are classified by complex line bundles on M.*

Proof. The proof is similar as the proof of theorem 2.1.
Notice that (cf. remark 1.9 (c) & 1.10 (b))

$$\begin{aligned} \varphi := \phi^c \times \rho_{\det} : \mathrm{Spin}^c(n) &\longrightarrow \mathrm{SO}(n) \times \mathrm{U}(1) \\ \lambda \otimes \sigma &\mapsto \big(\phi(\sigma), \lambda^2(\det \rho(\sigma))^2\big) \end{aligned} \tag{16}$$

is a well-defined group homomorphism with kernel $\mathbb{Z}_2$, thus we have the short exact sequence of non-Abelian sheaves

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathcal{S}pin^c(n) \xrightarrow{\varphi} \mathcal{SO}(n) \times S^1 \longrightarrow 1$$

and there is an induced exact sequence

$$\begin{array}{c} \longrightarrow H^1(M;\mathcal{S}pin^c(n)) \xrightarrow{\varphi^*} H^1(M;\mathcal{SO}(n)) \oplus H^1(M;S^1) \xrightarrow{\delta} H^2(M;\mathbb{Z}_2) \\ \| \\ H^1(M;\mathcal{SO}(n)) \oplus H^2(M;\mathbb{Z}) \end{array} \tag{17}$$

[11]One should be very careful that this $\mathcal{L}$ may *not* be $\det \$_M$! However, when $\dim M = 3$ they indeed coincide, see example 1.8.

[12]We will see from the proof of theorem 2.2 that P^c is actually a double cover of $P \times \mathcal{L}$.

[13]It is equivalent to say the third Stiefel-Whitney class of M vanishes $w_3(M) = 0$, see [6, Appendix B.13].

notice that the last morphism δ is $w_2 \oplus (r \circ c_1)$, where w_2 is taking the second Stiefel-Whitney class, c_1 is taking the first Chern class of a line bundle in $H^1(M; S^1)$, r is the modulo 2 reduction.

Now if $(g_{\alpha\beta}) \in H^1(M; \mathcal{SO}(n))$ admits a Spin^c-lifting $\left(\tilde{g}^c_{\alpha\beta}\right)$, then we see that its image under φ^* in (17) is exactly $g_{\alpha\beta} \oplus \rho_{\det}\left(\tilde{g}^c_{\alpha\beta}\right)$, they stand for the frame bundle P and the determinant line bundle $\mathcal{L}$, respectively. Therefore, by the exactness of (17), M admits a Spin^c structure one must have

$$w_2(M) + r \circ c_1(\mathcal{L}) = 0$$

which precisely means

$$w_2(M) = c_1(\mathcal{L}) \mod 2$$

Conversely, if a line bundle $\mathcal{L} = (h_{\alpha\beta}) \in H^1(M; S^1) = H^2(M; \mathbb{Z})$ satisfying the modulo 2 reduction of $c_1(\mathcal{L})$ is $w_2(M)$, then the image of $g_{\alpha\beta} \oplus h_{\alpha\beta} = P \oplus \mathcal{L}$ under δ is 0, hence again by the exactness of (17), there is a Čech cocycle $(\tilde{g}^c_{\alpha\beta}) \in H^1(M; \mathcal{S}pin^c(n))$ such that

$$\varphi^*\left(\tilde{g}^c_{\alpha\beta}\right) = \phi^c\left(\tilde{g}^c_{\alpha\beta}\right) \oplus \rho_{\det}\left(\tilde{g}^c_{\alpha\beta}\right) = g_{\alpha\beta} \oplus h_{\alpha\beta}$$

that is there exists [14] a Spin^c−structure with its spinor bundle $\$_M$ such that

$$\det \$_M = \mathcal{L}$$

As for the classification of Spin^c structures, we first fix a Spin^c structure $\left(\tilde{g}^c_{\alpha\beta}\right)$ and let $\$_M$ be its spinor bundle. Notice that for any line bundle $L = (\lambda_{\alpha\beta})$, the transition functions $\lambda_{\alpha\beta} \otimes \tilde{g}^c_{\alpha\beta}$ also define a Spin^c structure, since (cf. remark 1.9 (c))

$$\phi^c\left(\lambda_{\alpha\beta} \otimes \tilde{g}^c_{\alpha\beta}\right) = \phi^c\left(\tilde{g}^c_{\alpha\beta}\right) = g_{\alpha\beta}$$

and $\lambda_{\alpha\beta} \otimes \tilde{g}^c_{\alpha\beta}$ obviously satisfies the cocycle condition [15]. Let $\$'_M$ be the spinor bundle associated to the twisted Spin^c structure, we see that its transition functions are (cf. remark 1.10 (c))

$$\rho^c\left(\lambda_{\alpha\beta} \otimes \tilde{g}^c_{\alpha\beta}\right) = \lambda_{\alpha\beta}\rho^c(\tilde{g}^c_{\alpha\beta})$$

[14] We can see that the existence of such a Spin^c−structure is *not* unique, since by the exactness of (17), it is unique if and only if $\delta = w_2 \oplus r \circ c_1$ is surjective.

However, this is not always the case, for example, if (on a Spin^c manifold) $H^2(M; \mathbb{Z})$ has a 2-torsion, i.e. there exists a line bundle $L = (\lambda_{\alpha\beta})$ with $L^2 = L \otimes L$ is trivial, then for a given Spin^c structure $\tilde{g}^c_{\alpha\beta}$ (whose determinant line bundle is denoted by $\mathcal{L}$), we have a new Spin^c structure obtained by twisted by L, namely $\lambda_{\alpha\beta} \otimes \tilde{g}^c_{\alpha\beta}$, its associated determinant line bundle is denoted by $\mathcal{L}'$. These two Spin^c structures not coincide unless L is trivial, but we can find that $\mathcal{L} = \mathcal{L}'$, since the transition function on $\mathcal{L}'$ is (cf. remark 1.10 (b))

$$\rho_{\det}\left(\lambda_{\alpha\beta} \otimes \tilde{g}^c_{\alpha\beta}\right) = \lambda^2_{\alpha\beta}\rho_{\det}\left(\tilde{g}^c_{\alpha\beta}\right)$$

and $L \otimes L$ is trivial implies $\lambda^2_{\alpha\beta} = 1$, hence $\mathcal{L}$ and $\mathcal{L}'$ have same transition function.

What's more, one can show by the universal coefficient theorem that δ is surjective (and hence the Spin^c structure is uniquely determined by its determinant line bundle) if and only if $H^2(M; \mathbb{Z})$ has no 2-torsions.

[15] The Spin^c structure obtained by this way is called *twisted by a line bundle L*

hence $\not{S}'_M = \not{S} \otimes L$. Therefore, the twisted Spin^c structure coincides with the original one if and only if

$$\not{S}_M = \not{S}'_M = \not{S}_M \otimes L$$

which if and only if L is trivial, hence all Spin^c structures $\mathcal{S}(M)$, after picking a frame point $(\tilde{g}^c_{\alpha\beta})$, is an affine space modeled on $H^2(M;\mathbb{Z})$. ♣

We refer to [6, Appendix D], [13, Theorem 5.8] or this note for some other proofs

Example 2.3. (1) Every spin manifold is a Spin^c manifold, since $\text{Spin}(n) \subset \text{Spin}^c(n)$, we can extend its spin structure to a Spin^c structure by twisting by any line bundle L.

Hence in particular every oriented compact 3-manifolds are Spin^c manifolds. But different from the spin case (see Example 2.2), neither the spinor bundle $\not{S}$ nor the $\det \not{S}$ associated to a Spin^c structure are necessarily to be trivial unless it was twisted by a trivial line bundle, since $\text{Spin}^c(3) \cong \text{U}(2)$.

As a special case, let $\not{S}_M$ be the spinor bundle associated to the spin structure, then the spinor bundle of the "trivial" Spin^c structure (that is by twisting with a trivial line bundle) is simply $\not{S}_M \otimes L$, where L is a trivial line bundle.

Moreover, the determinant line bundle associated to this trivial Spin^c structure is $\mathcal{L} = \det \not{S}_M$. Therefore, by theorem 2.2, we have

$$0 = w_2(M) = c_1(\not{S}_M) \mod 2$$

(2) $\mathbb{CP}^2$ is a Spin^c manifold. It is not the only case, since every oriented compact 4-manifold admits a Spin^c structure. See theorem 2.3.

Remark 2.2.
- If a Spin^c manifold (M, g) is also spin, then we can *divide* its Spin^c structure by its spin structure, that yields a *well-defined* line bundle L_0. If we denoted $\not{S}^c_M$ and $\not{S}_M$ to be the spinor bundles associated to the Spin^c and spin structures respectively, then we see that

$$\not{S}^c_M = \not{S}_M \otimes L_0$$

hence sometimes, Spin^c structure is also called the *twisted spin structure by a line bundle* L_0. As a consequence, we have

$$\det \not{S}^c_M = \det(\not{S}_M \otimes L_0) = \det \not{S}_M \otimes L_0^{2^{[\frac{n}{2}]}}$$

hence we have their first Chern classes

$$c_1\left(\not{S}^c_M\right) = c_1(\not{S}_M) + 2^{\left[\frac{n}{2}\right]} c_1(L_0) = 0 \mod 2$$

The determinant line bundle of this Spin^c structure is L_0^2

- If a manifold is *just* Spin^c, the notation $\not{S}_M \otimes L_0$ is not well-defined. But in some literatures (cf. [3]), they will still use this notation to present the spinor bundle, and L_0 is called the *virtual line bundle*.

Theorem 2.3. *Every oriented compact 4-manifold* (M, g) *admits a* Spin^c *structure.*

Proof. We first claim that the Spin^c structure exists for a *simply connected* (M, g). Since from the short exact sequence of coefficients:

$$1 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 1$$

we have

$$\cdots \to H^2(M;\mathbb{Z}) \longrightarrow H^2(M;\mathbb{Z}_2) \xrightarrow{\delta} H^3(M;\mathbb{Z}) = 0$$

where we have by Poincaré duality that $H^3(M;\mathbb{Z}) \cong H_1(M;\mathbb{Z}) = 0$. Thus $w_2(M)$ has a lifting if and only if its image under δ is zero, which is exactly the case.

Now if M is not simply connected, thanks to the compactness, we can choose a good cover $\{U_\alpha\}$ together with a decomposition of the unit $f_\alpha : U_\alpha \longrightarrow \mathbb{R}$.

Since $U_{\alpha\beta\gamma} = U_\alpha \cap U_\alpha \cap U_\gamma$ is simply connected, every cocycle $\eta_{\alpha\beta\gamma} \in H^2(U_{\alpha\beta\gamma};\mathbb{Z}_2)$ which presents $w_2(M)$ can be lifted into some $\tilde{\eta}_{\alpha\beta\gamma} \in H^2(U_{\alpha\beta\gamma};\mathbb{Z})$, we will use this piecewise lifting to construct a Spin$^c(4)$-lifting of the frame bundle $P = (g_{\alpha\beta})$.

Let

$$h_{\alpha\beta} = \exp\left(\sqrt{-1}\pi \sum_\gamma f_\gamma \tilde{\eta}_{\alpha\beta\gamma}\right) : U_\alpha \cap U_\beta \longrightarrow \mathrm{U}(1)$$

by computation we find that

$$h_{\alpha\beta} h_{\beta\gamma} h_{\gamma\alpha} = \eta_{\alpha\beta\gamma}$$

Also, if we denoted by $\tilde{g}_{\alpha\beta}$ a Spin(4) lifting of the cocycle $g_{\alpha\beta}$, we know that the obstructions are exactly the second Stiefel-Whitney class

$$\eta_{\alpha\beta\gamma} = \tilde{g}_{\alpha\beta} \tilde{g}_{\beta\gamma} \tilde{g}_{\gamma\alpha}$$

hence

$$h_{\alpha\beta} \tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Spin}^c(n)$$

defines a Čech cocycle which is a lifting of $g_{\alpha\beta}$. ♣

Remark 2.3. For a 4-manifold, recall that by Example 1.7, its Spin^c structure has the form

$$\tilde{g}^c_{\alpha\beta} = \begin{pmatrix} \lambda_{\alpha\beta} A_{\alpha\beta+} & O \\ O & \lambda_{\alpha\beta} A_{\alpha\beta-} \end{pmatrix} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Spin}^c(4)$$

hence by example 1.8 the transition function of the determinant line bundle $\mathcal{L}$ is

$$\lambda^2_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{U}(1)$$

Note that its Spin^c-spinor bundle has the form $\$_+ \oplus \$_-$ (cf. Example 1.8), we see that

$$\mathcal{L} = \det \$_+ = \det \$_-$$

and

$$\mathcal{L} \otimes \mathcal{L} = \det\left(\$_+ \oplus \$_-\right)$$

2.3 Structures on Spinor Bundles

Spinor bundles $\$_M$ will have three natural structures. First each fiber endows with a Clifford action, which makes it a bundle of Clifford module. Then there is a natural connection induced from the Levi-Civita connection, which will be called the spin connection. Finally, these two structures make $\$_M$ becomes a Dirac bundle, hence there is a natural Dirac operator defined on it.

2.3.1 Bundles of Clifford Module

Same as definition 1.2, we can globalize the notion of Clifford modules to a geometric object.

Definition 2.7. A (complex) vector bundle $E \longrightarrow M$ is called a bundle of Clifford module, if it endows with an action of TM:

$$C\ell : TM \otimes E \longrightarrow E$$

such that $v \cdot (v \cdot s) = -g(v, v)s = -|v|^2 s$. ♣

So a bundle of Clifford module is just a vector bundle with each fiber endows with a Clifford multiplication. Some examples can be constructed by globalizing example 1.2 and 1.3.

Example 2.4.
(1) If (M, g) is a spin or Spin^c manifold, then its spinor bundle $\$_M$ associated with the spin or Spin^c structure is a bundle of Clifford module.

(2) In particular if $\dim M = 4$, then its spinor bundle (with respect to a spin or Spin^c structure) splits as $\$_+ \oplus \$_-$. Choose a local frame $e_1, \ldots, e_4$ of TM, then the Clifford multiplication of e_i's on the fiber (ψ_+, ψ_-) can be given by (see (4) and Example 1.3)

$$\begin{aligned} e_1 \cdot (\psi_+, \psi_-) &:= (\psi_+, \psi_-) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; e_2 \cdot (\psi_+, \psi_-) = (\psi_+, \psi_-) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ e_3 \cdot (\psi_+, \psi_-) &:= (\psi_+, \psi_-) \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \; e_3 \cdot (\psi_+, \psi_-) := (\psi_+, \psi_-) \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \end{aligned} \tag{18}$$

where i, j, k are the imaginary unit in $\mathbb{H}$ which should be presented by 2×2 complex matrices, see (14). We can also see that the Clifford multiplication changes the chirality.

(3) Similarly, if $\dim M = 3$, the Clifford action of a local frame e_1, e_2, e_3 on $\psi \in \$$ is given by multiplication with i, j, k respectively, where again i, j, k should be presented by (14).

(4) The exterior bundle $\bigwedge T^*_{\mathbb{C}} M$ is a bundle of Clifford module, where the Clifford multiplication can be given by (4).

Remark 2.4. (i) Identifying $TM \cong T^*M$ by the Riemannian metric, for a cotangent vector ω in T^*M, its covector ω^* can be expressed by the local frame $e_1, ..., e_n$ of TM by

$$\omega^* = \sum_{i=1}^{n} \omega(e_i)e_i$$

hence if E is a bundle of Clifford module, then there is also an action

$$C\ell : T^*M \otimes E \longrightarrow E$$

given by

$$C\ell(\omega^* \otimes s) = \sum_{i=1}^{n} \omega(e_i)e_i \cdot s$$

(ii) If E is a bundle of Clifford module, by extending the Clifford multiplication of T^*M, there is also an action of the exterior bundle $\bigwedge T^*_{\mathbb{C}}M$, which maps such as for ω a 2−form,

$$C\ell : \omega \otimes s \mapsto \sum_{i,j} \omega(e_i, e_j)e_i \cdot e_j \cdot s$$

(iii) If E is a bundle of Clifford module, by extending the Clifford multiplication of TM, there is also an action of the exterior bundle $\bigwedge TM$ (and hence also $\bigwedge T^*M$), which will still be called the Clifford multiplication, and it maps as

$$C\ell : e_1 \wedge \cdots \wedge e_k \otimes s \mapsto e_1 \cdot \cdots \cdot e_k \cdot s$$

(iv) In particular, by (18) in Example 2.4, we can compute that if $\dim M = 4$, then the self-dual part $\bigwedge^2_+ T^*M$ acts trivially on $\$_-$, that is for self-dual 2-form ω^+,

$$\omega^+ \cdot (\psi_+, \psi_-) = \omega^+ \cdot (\psi_+, 0) = (\omega^+ \cdot \psi_+, 0)$$

and dually, the anti-self-dual part acts trivially on $\$_+$.

Moreover, by computation, we find that this Clifford action

$$C\ell : \bigwedge\nolimits^2_+ T^*M \longrightarrow \mathrm{End}\left(\$_+\right)$$

is faithful, whose is image is just $\mathfrak{su}\left(\$_+\right)$, hence $\bigwedge^2_+ T^*M$ can be identified with $\mathfrak{su}(\$_+)$.

2.3.2 Connections on Spinor Bundles

Now we assume (M, g) is a spin or Spin^c manifold, then on the spinor bundle $\$_M$ associated to a *chosen* Spin^c or spin structure, we can canonically construct a connection ∇ on it, called the *spin connection*.

- First let me recall that if there is a local diffeomorphism (for example covering map) between two Lie groups $\phi : G \longrightarrow H$, then the tangent $(d\phi)_1$ identifies their Lie algebras $\mathfrak{g} \cong \mathfrak{h}$. Now if ω is a connection on a principal H-bundle $P_H \longrightarrow M$ (which is locally an $\mathfrak{h}$-valued 1-form on M), and let P_G be a principal G-bundle which lifts P_H along ϕ, then ω uniquely [16] induces a connection $\tilde{\omega}$ on P_G by

$$(d\phi)_1 \tilde{\omega} = \omega$$

- Now, we assume (M, g) has a spin structure $\tilde{P}$, of course it is a Spin(n)-principal bundle which lifts the SO(n)-frame bundle P along ϕ. Thus there exists a unique connection $\tilde{\omega}$ on $\tilde{P}$ induced by the Levi-Civita connection ω on P.

  Notice that locally, the Levi-Civita connection $\omega = (\omega_{ij})$ is an $\mathfrak{so}(n)-$valued 1-form, it can be write as

$$\omega = (\omega_{ij}) = \sum_{i<j} \omega_{ij} e_i \wedge e_j$$

  where $e_i \wedge e_j$'s are basis of $\mathfrak{so}(n) \cong \bigwedge^2 \mathbb{R}^n$ (see theorem 1.2).

  Hence it *uniquely induces* a connection $\tilde{\omega}$ on $\tilde{P}$ by the tangent map of the real representation

$$(d\phi)_1 : \mathfrak{spin}(n) \cong \mathfrak{so}(n) \longrightarrow \mathfrak{so}(n)$$

  which we claim locally it is

$$\tilde{\omega} = \frac{1}{2} \sum_{i<j} \omega_{ij} e_i \wedge e_j$$

  Indeed, we can compute by theorem 1.2:

$$\begin{aligned}
(d\phi)_1 \left(\frac{1}{2} \sum_{i<j} \omega_{ij} e_i \wedge e_j \right) &= \sum_{i<j} \omega_{ij} (d\phi)_1 \left(\frac{e_i \wedge e_j}{2} \right) \\
&= \sum_{i<j} \omega_{ij} \left. \frac{d}{dt} \right|_{t=0} \phi \left(\exp \left(\frac{t}{2} (e_i \wedge e_j) \right) \right) \\
&= \sum_{i<j} \omega_{ij} \left. \frac{d}{dt} \right|_{t=0} \phi \left(\cos \frac{t}{2} + \sin \frac{t}{2} e_i \cdot e_j \right) \\
&= \sum_{i<j} \omega_{ij} \left. \frac{d}{dt} \right|_{t=0} \exp(t(e_i \wedge e_j)) \\
&= \sum_{i<j} \omega_{ij} e_i \wedge e_j = \Omega
\end{aligned}$$

[16] The uniqueness is due to the fact that ϕ is a local diffeomorphism and hence $(d\phi)_1$ is an isomorphism.

- Moreover, the induced connection $\tilde{\omega}$ will induce a connection ∇ on its adjoint bundle $\$_M$
$$\nabla : \Gamma(\$_M) \longrightarrow \Gamma(\$_M \otimes T^*M)$$
by
$$(d\rho)_1 : \mathfrak{spin}(n) \cong \mathfrak{so}(n) \longrightarrow \mathfrak{gl}(\$_M)$$
it can be write as
$$\begin{aligned}\nabla\psi = d\psi + ((d\rho)_1\tilde{\omega})(\psi) &= d\psi + \frac{1}{2}\sum_{i<j}\omega_{ij}e_i \cdot e_j \cdot \psi \\ &= d\psi + \frac{1}{4}\sum_{i,j}\omega_{ij}e_i \cdot e_j \cdot \psi\end{aligned} \tag{19}$$
We call this induced connection ∇ the **spin connection** on $\$_M$ induced by a chosen spin structure.
- Now, if (M, g) is a Spin^c manifold with a Spin^c structure P^c, let $\mathcal{L}$ be the associated line bundle. Then from (16) we know that P^c lifts $P \times \det \mathcal{L}$ along
$$\begin{aligned}\varphi := \phi^c \times \rho_{\det} : \mathrm{Spin}^c(n) &\longrightarrow \mathrm{SO}(n) \times \mathrm{U}(1) \\ \lambda \otimes \sigma &\mapsto (\phi(\sigma), (\rho_{\det}(\lambda \otimes \sigma)))\end{aligned}$$
Let ω be the Levi-Civita connection on P as before. In order to induce a connection on P^c, we also need to fix a U(1)-connection A on $\mathcal{L}$.
Write $A = d + \sqrt{-1}a$ where $a \in \Omega^1(M;\mathbb{R})$, which is an U(1)-connection on $\mathcal{L}$, hence now, by the same method as before, $(d\varphi)_1$ will induce a connection ∇_A on $\$_M$, it write as (by remark 1.9 (d))
$$\begin{aligned}\nabla_A\psi &= d\psi + \frac{1}{2}\left(a + \sum_{i<j}\omega_{ij}e_i \cdot e_j\right) \cdot \psi \\ &= d\psi + \frac{1}{2}a \cdot \psi + \frac{1}{4}\sum_{i,j}\omega_{ij}e_i \cdot e_j \cdot \psi\end{aligned} \tag{20}$$
where $a \cdot \psi$ is the Clifford multiplication by T^*M, see remark 2.4. This ∇_A will still be called the **spin connection** on $\$_M$ although it is induced by a Spin^c structure.

By our construction, the spin connections ∇ or ∇_A will satisfy the following property:

Theorem 2.4. *(i) The spin connection ∇ is* compatible *with the Levi-Civita connection D on TM, that is for any $v \in \Gamma(TM)$ and $\psi \in \Gamma(\$_M)$*
$$\nabla(v \cdot \psi) = (Dv) \cdot \psi + v \cdot (\nabla\psi) \tag{21}$$
where $\cdot$ stands for the Clifford multiplication.

(ii) *The spin connection is also compatible with the Hermitian metric $\langle \cdot, \cdot \rangle$ on $\$_M$ in the sense that for any $v \in \Gamma(TM)$ and $\psi_1, \psi_2 \in \Gamma(\$_M)$, we have*

$$\frac{\partial \langle \psi_1, \psi_2 \rangle}{\partial v} = \langle \nabla_v \psi_1, \psi_2 \rangle + \langle \psi_1, \nabla_v \psi_2 \rangle \tag{22}$$

The proof of this two results can be found in [6, Prop 4.4 & 4.11] (or leave as an exercise).

Then it is natural to study the curvature of the spin connections. In what follows we denote

$$\Omega = D^2 = d\omega + \omega \wedge \omega \in \Omega^2(M; \mathrm{End}(TM))$$

to be the curvature of the Levi-Civita connection $D = d + \omega$, and denote $R = \nabla^2$ or $R_A = \nabla_A^2$ to be the curvature of spin connections, which are $\mathrm{End}(\$_M)$-valued 2-forms, as a notation convention, we use $R_{e_i e_j} = R(e_i, e_j)$ to be the value [17] of a curvature R at $e_i \wedge e_j \in \bigwedge^2 T_x M$.

Also recall that from Riemannian geometry, the *scalar curvature* $s_g(x)$ of a Riemannian manifold (M, g) is defined by

$$\begin{aligned} s_g(x) := \sum_{i,j} \Omega_{ijij}(x) &= \sum_{i,j} g_x\left(\Omega_{e_i e_j}(e_i), e_j\right) \\ &= \sum_{i,j} \mathrm{Ric}_g(e_i, e_j) \end{aligned}$$

where e_i's are local frames of TM at x.

Lemma 2.1. *Let $e_1, \ldots, e_n$ be a* geodesic *local orthonormal frame of TM, that is $D_{e_i} e_j = 0$, then*

(a) *If M is a spin manifold with the spinor bundle $\$_M$, then for any $\psi \in \Gamma(\$_M)$, the curvature R satisfies*

$$\frac{1}{2} \sum_{i,j} e_i \cdot e_j R_{e_i e_j}(\psi) = \frac{s_g}{4} \psi$$

(b) *If M is a* Spin^c *manifold with the spinor bundle $\$_M$, and let $A = d + a$ be a connection onvits determinant line bundle $\mathcal{L}$, then for any $\psi \in \Gamma(\$_M)$, the curvature R_A satisfies*

$$\frac{1}{2} \sum_{i,j} e_i \cdot e_j R_{A_{e_i e_j}}(\psi) = \frac{s_g}{4} \psi + \frac{F_A}{2} \cdot \psi$$

where $F_A = da$ is the curvature of A, the $\cdot$ means the Clifford multiplication (see remark 2.4 (iii)).

The proof can be done by a straightforward computation of R and R_A, which can be found in [3, pp. 63] (or leave as an exercise).

[17] A two form ω is a form $\omega_x : \bigwedge^2 T_x M \longrightarrow \mathbb{R}$ which is smooth as x varies.

2.3.3 Dirac Bundles and Dirac Operators

The spin connection give a new structure on $\$_M$, called the Dirac bundle.

Definition 2.8 (Dirac Bundle)**.** A bundle of Clifford module $E \longrightarrow M$ is called a *Dirac bundle* if there is a connection ∇ on E, called the *Dirac connection*, such that it is compatible with the Levi-Civita connection D on TM:

$$\nabla(v \cdot s) = (Dv) \cdot s + v \cdot (\nabla s)$$

where $v \in \Gamma(TM)$, $s \in \Gamma(E)$ and the $\cdot$ is the Clifford multiplication. ♣

What makes Dirac special is that there exists a natural operator on $\Gamma(E)$:

Definition 2.9 (Dirac Operator)**.** If E is Dirac bundle with a Dirac connection ∇, then the operator

$$\mathcal{D} : \Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{C\ell} \Gamma(E)$$

is called the *Dirac operator*. We call $\mathcal{D}^2$ the *Dirac Laplacian operator*. ♣

Remark 2.5. Let $e_1, ..., e_n$ be a local frame of TM, then Dirac operator has a local expression

$$\mathcal{D}s = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i} s$$

what's more, one can show that this local expression is independent of the choice of the frames. ♣

Example 2.5. (a) If (M, g) is a Spin^c or spin manifold, then the spinor bundle $\$_M$ is a Dirac bundle, the Dirac connection is the spin connection, since by (21) they satisfy the compatibility condition. As a notation convention the Dirac operator on $\$_M$ will be denoted by $\not{D}$ or $\not{D}_A$ if we refer to a Spin^c structure.

(b) In particular, if $\dim M = 4$, the spinor bundle splits as $\$_+ \oplus \$_-$, recall that the Clifford action on $\$_+ \oplus \$_-$ changes the chirality (cf. Example 2.4 (18)), hence the Dirac operator $\not{D}$ also splits as $\begin{pmatrix} & \not{D}^- \\ \not{D}^+ & \end{pmatrix}$, where

$$\not{D}_\pm : \Gamma\left(\$_\pm\right) \longrightarrow \Gamma\left(\$_\mp\right)$$

which changes the chirality.

(c) For a special case, if (M, g) is just $\mathbb{R}^4$ with the standard Euclidean metric, then it has both unique spin and Spin^c structures [18]. We will compute $\not{D}$ or $\not{D}_A$ concretely.

For the spin structure, let's compute $\not{D}^+$ first. Recall that $\$_\pm$ can be viewed as the trivial quaternionic line bundle $\mathbb{R}^4 \times \mathbb{H}$, hence a section $f \in \Gamma(\$_\pm)$ can be viewed

[18] They actually coincide.

as a quaternionic-valued function on M. Since the Levi-Civita connection is the trivial one, the spin connection is simply taking directional derivative, hence by (18) we have

$$\begin{aligned}
\not{D}^+ f &= \mathrm{C}\ell(\nabla f) = \mathrm{C}\ell\left(\sum_{i=0}^{3} \frac{\partial f}{\partial x_i} \otimes dx_i\right) \\
&= \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1} i + \frac{\partial f}{\partial x_2} j + \frac{\partial f}{\partial x_3} k
\end{aligned}$$

hence we see that $\not{D}^+$ is just the quaternionic Cauchy-Riemann operator.

Similarly, by (7) we have

$$\not{D}^- = -\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} i + \frac{\partial}{\partial x_2} j + \frac{\partial}{\partial x_3} k$$

and it is clearly that $\not{D}^2 = \begin{pmatrix} \not{D}^- \not{D}^+ & 0 \\ 0 & \not{D}^- \not{D}^+ \end{pmatrix}$ is the usual Laplacian on $\mathbb{R}^4$.

Let $A = d + \sqrt{-1}a$ be a connection on $\mathcal{L} = \det \not{S}_+ = \det \not{S}_-$, then by (20), $\not{D}_A$ is simply

$$\not{D}_A f = \not{D} f + \frac{\sqrt{-1}a \cdot f}{2}$$

(d) When $M = \mathbb{R}^3$, the Dirac operator is just

$$\not{D} = \frac{\partial}{\partial x_1} i + \frac{\partial}{\partial x_2} j + \frac{\partial}{\partial x_3} k$$

Again, $\not{D}^2$ is the usual Laplacian on $\mathbb{R}^3$, hence the Dirac operator can be understood as the "square root" of the Laplacian.

(e) If $M = \mathbb{R}^2$, then $\not{D}^\pm$ are just the usual Cauchy-Riemann operator.

(f) Recall that the Levi-Civita connection can be extended to T^*M and $\bigwedge T^*_{\mathbb{C}}M$, which will still be denoted by D. Hence by (5), the exterior bundle $\bigwedge T^*_{\mathbb{C}}M$ is a Dirac bundle with the Levi-Civita connection D as its Dirac connection.

Moreover, the Dirac operator is just the Hodge operator $\mathcal{D} = d+\delta$, where $\delta = -{*}d{*}$ is the Hodge codifferential.

To see this, we notice that for the induced Levi-Civita connection on T^*M, one has

$$\begin{cases} \delta\omega = \sum_i \iota_{e_i} D_{e_i}\omega \\ d\omega = -\sum_i g(e_i, \cdot) \wedge D_{e_i}\omega \end{cases}$$

by remark 2.5 and (5), the result follows by a straightforward computation:

$$\begin{aligned}
\mathcal{D}\omega &= \sum_i e_i \cdot D_{e_i}\omega = \sum_{i=1} \left(\iota_{e_i} D_{e_i}\omega - g(e_i, \cdot) \wedge D_{e_i}\omega\right) \\
&= (d + \delta)\omega
\end{aligned}$$

Remark 2.6. For $\not{D}_A$, it is obviously that by (20), we have that if $a \in \Omega^1(M; \sqrt{-1}\mathbb{R})$ then $A + a$ is another connection on the determinant line bundle $\mathcal{L}$, then

$$\not{D}_{A+a}\psi = \not{D}_A\psi + \frac{a \cdot \psi}{2}$$

2.4 Properties of Dirac Operators

In this subsection, I will introduce 3 main properties of the Dirac operator. In this section, the Dirac operators will be the ones on a spinor bundle $\not{S}_M$ for M with a spin or Spin^c structure.

2.4.1 Formal Self-Adjointness

Let $\langle \cdot, \cdot \rangle_x$ be the Spin^c or spin invariant Hermitian metric on each fiber of $\not{S}_M$, then choose a volume form (an orientation) dV on M, we can define an L^2-inner product $(\cdot, \cdot)_{L^2}$ on $\Gamma(\not{S}_M)$: for any two $\psi_1, \psi_2 \in \Gamma(\not{S}_M)$, define

$$(\psi_1, \psi_2)_{L^2} := \int_M \langle \psi_1, \psi_2 \rangle_x dV$$

Theorem 2.5. *The Dirac operator $\not{D}$ or $\not{D}_A$ is formal-self adjoint. That is*

$$(\not{D}\psi_1, \psi_2)_{L^2} = (\psi_1, \not{D}\psi_2)_{L^2}$$

the formula for $\not{D}_A$ is similar.

Proof. Let $e_1, ..., e_n$ be a geodesic orthonormal frame on TM, that is $D_{e_i}e_j = 0$, then by (22) and remark 2.5 we have

$$\begin{aligned}
(\not{D}\psi_1, \psi_2)_{L^2} &= \int_M \left\langle \sum_{i=1}^n e_i \cdot \nabla_{e_i}\psi_1, \psi_2 \right\rangle_x dV = -\int_M \sum_{i=1}^n \langle \nabla_{e_i}\psi_1, e_i \cdot \psi_2 \rangle_x dV \\
&= -\int_M \sum_{i=1}^n \left(\frac{\partial \langle \psi_1, e_i \cdot \psi_2 \rangle_x}{\partial e_i} - \langle \psi_1, \nabla_{e_i}(e_i \cdot \psi_2) \rangle_x \right) dV \\
&= -\int_M \sum_{i=1}^n \left(\frac{\partial \langle \psi_1, e_i \cdot \psi_2 \rangle_x}{\partial e_i} - \langle \psi_1, e_i \cdot \nabla_{e_i}\psi_2 \rangle_x \right) dV \\
&= (\psi_1, \not{D}\psi_2)_{L^2} - \int_M \sum_{i=1}^n \frac{\partial \langle \psi_1, e_i \cdot \psi_2 \rangle_x}{\partial e_i} dV
\end{aligned}$$

The second term in the last line looks likes a divergent of some vector field. Indeed, define $X \in \Gamma(T_{\mathbb{C}}M)$ by

$$X(x) = \sum_{i=1}^n \langle \psi_1, e_i \cdot \psi_2 \rangle_x \cdot e_i$$

then we find

$$\mathrm{div} X = \sum_{i=1}^{n} \frac{\partial g(e_i, X)}{\partial e_i} = \sum_{i=1}^{n} \frac{\partial \langle \psi_1, e_i \cdot \psi_2 \rangle_x}{\partial e_i}$$

hence by Cartan magic formula and Stokes theorem, due to the compactness of M we have

$$\begin{aligned}\int_M \sum_{i=1}^{n} \frac{\partial \langle \psi_1, e_i \cdot \psi_2 \rangle_x}{\partial e_i} dV &= \int_M \mathrm{div} X dV = \int_M \mathcal{L}_X dV \\ &= \int_M d\left(\iota_X dV\right) - \int_M \iota_X d(dV) \\ &= 0\end{aligned}$$

the proof of $\not{D}_A$ is similar. ♣

2.4.2 Weitzenböck Formula

Another useful property of Dirac operator is a Bochner type formula, called *Weitzenböck formula*. To formulate this formula, we need to introduce what is a connection Laplacian first.

Recall that any connection ∇ on E can be extended to $E \otimes \bigwedge^k T^*M$ by

$$\nabla \omega \otimes s = (D\omega) \otimes s + (-1)^{\deg \omega} \omega \otimes \nabla s$$

where D is the extended Levi-Civita connection $\bigwedge^k T^*M$, see example 2.5 (f). We can regard $\Gamma\left(E \otimes \bigwedge^k T^*M\right)$ as $\Omega^k(M; E)$, i.e. k-forms with values in E, then the extended ∇ can be seen as a map

$$d_\nabla : \Omega^k(M; E) \longrightarrow \Omega^{k+1}(M; E)$$

in particular, there is a "de Rham-like complex"

$$\cdots \to \Omega^{k-1}(M; E) \xrightarrow{d_\nabla} \Omega^k(M; E) \xrightarrow{d_\nabla} \Omega^{k+1}(M; E) \longrightarrow \cdots$$

this is indeed a complex if and only if ∇ is flat.

Like what we do in Hodge theory, we can define a codifferential

$$d_\nabla^* = (-1)^{n(k-1)+1} * \circ d_\nabla \circ * : \Omega^k(M; E) \longrightarrow \Omega^k(M; E)$$

Definition 2.10 (Connection Laplacian)**.** Let (E, ∇) be a bundle over (M, g) with a connection ∇, the operator

$$\nabla^* \nabla : \Gamma(E) \longrightarrow \Gamma(E \otimes T^*M) \xrightarrow{\nabla^* := d_\nabla^*} \Gamma(E)$$

is called the *connection Laplacian*. ♣

The connection Laplacian has the following properties.

Theorem 2.6. *(1) It has a local expression: let $e_1, \ldots, e_n$ be a group of orthonormal local frame on TM, then*

$$\nabla^*\nabla s = -\sum_{i=1}^{n} \left(\nabla_{e_i}\nabla_{e_i}s - \nabla_{D_{e_i}e_i}s\right) \tag{23}$$

and this expression is independent of the choice of the frames.

(2) $\nabla^*\nabla$ is formal self-adjoint, that is if there is a metric $\langle\cdot,\cdot\rangle$ on E compatible with ∇ in sense of (22), then

$$\int_M \langle\nabla^*\nabla s_1, s_2\rangle_x \mathrm{vol}_M = \int_M \langle\nabla s_1, \nabla s_2\rangle_x \mathrm{vol}_M = \int_M \langle s_1, \nabla^*\nabla s_2\rangle_x \mathrm{vol}_M$$

The proof can be found in [2, 14] (or leave as an exercise).

Remark 2.7. Notice that in the local expression (23), the term $-\sum \nabla_{e_i}\nabla_{e_i}$ already looks likes the usual Laplacian, but this form *depends* on the choice of basis, by minus the second will kill this dependency. Obviously, if we choose e_i's to be the geodesic frame such that $D_{e_i}e_j = 0$, then we see that the second term will be annihilated.

Theorem 2.7 (Weitzenböck Formula)**.** *Let (M, g) be a spin or* Spin^c *bundle, let ∇ or ∇_A be the spin connection on the spinor bundle $\$_M$.*

(1) For $\not{D}$, we have

$$\not{D}^2\psi = \nabla^*\nabla\psi + \frac{s_g}{4}\psi$$

(2) For $\not{D}_A$ we have

$$\not{D}_A^2\psi = \nabla_A^*\nabla_A\psi + \frac{s_g}{4}\psi + \frac{F_A\cdot\psi}{2}$$

Proof. For the first formula, since (23) is independent of choice of basis, we can take $e_1, \ldots, e_n$ be the geodesic frame local frame on TM, hence by lemma 2.1 and remark 2.5 we have

$$\begin{aligned}
\not{D}^2\psi &= \sum_{i=1}^{n} e_i\cdot\nabla_{e_i}\left(\sum_{j=1}^{n} e_j\cdot\nabla_{e_j}\psi\right) = \sum_{i,j} e_i\cdot e_j\cdot\nabla_{e_i}\nabla_{e_j}\psi \\
&= -\sum_{i=1}^{n}\nabla_{e_i}\nabla_{e_i}\psi + \frac{1}{2}\sum_{i\neq j} e_i\cdot e_j\cdot\left(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i}\right)\psi \\
&= \nabla^*\nabla\psi + \frac{1}{2}\sum_{i,j} e_i\cdot e_j\cdot R_{e_ie_j}(\psi) \\
&= \nabla^*\nabla\psi + \frac{s_g}{4}\psi
\end{aligned}$$

Similarly, by lemma 2.1 (b) we can prove the second formula. ♣

Remark 2.8. If $\dim M = 4$ is a Spin^c manifold, and let A be a connection on the determinant line bundle $\mathcal{L} \cong \det \$_+ = \det \$_-$, recall that from example 2.5, the Dirac operator on $\$_+ \oplus \$_-$ splits as $D\!\!\!/^{\pm}_A$, then for $\psi \in \Gamma(\$_+)$, the Weitzenböck formula reads as

$$D\!\!\!/^{-}_A D\!\!\!/^{+}_A \psi = \nabla^*_A \nabla^*_A \psi + \frac{s_g}{4}\psi + \frac{1}{2} F^+_A \cdot \psi \tag{24}$$

where $F^+_A \in \Omega^2_+ \left(M; \sqrt{-1}\mathbb{R}\right)$ is the self-dual part of the curvature F_A, it is due to remark 2.4 (iv) that an anti-self dual 2-form acts trivially on $\$_+$. ♣

2.4.3 Atiyah-Singer Index Theorem

From now on, we assume (M, g) is compact oriented Riemannian manifold of *even-dimension* (and in particular we will interest in the case $\dim M = 4$). Recall that the Dirac operator $D\!\!\!/$ splits as

$$D\!\!\!/^{\pm} : \Gamma\left(\$_{M_\pm}\right) \longrightarrow \Gamma\left(\$_{M_\mp}\right)$$

Definition 2.11. The *index* of the Dirac operator is defined as

$$\mathrm{Ind} D\!\!\!/ := \mathrm{Ind} D\!\!\!/^{+} := \dim\ker D\!\!\!/^{+} - \dim\ker D\!\!\!/^{-}$$

the index of $D\!\!\!/_A$ is similar. ♣

The index of the Dirac operator is an analytic invariant, it measures how big is the space of solutions of the linear elliptic PDE $D\!\!\!/\psi = 0$, the importance is that it is also a topological invariant.

Theorem 2.8 (Atiyah-Singer)**.** *If M is a $2n$-dimensional* Spin^c *manifold with $\mathcal{L}$ the determinant line bundle, then for any connection A on $\mathcal{L}$, the index of $D\!\!\!/_A$ is a topological invariant:*

$$\mathrm{Ind}\, D\!\!\!/_A = \int_M \hat{A}(M) \smile \mathrm{ch}\left(\mathcal{L}\right)$$

where $\hat{A}$ is called the Hirzebruch class, $\mathrm{ch}(\mathcal{L})$ *stands for the total Chern class.*

For more details about Hirzebruch classes and Chern-Weil theory can be found in [14].

The following example is the index of the Hodge operator $d+\delta$ on the exterior bundle $\bigwedge T^*M$, although it is not a spinor bundle, the example is good for us to understand what does the index measure.

Example 2.6. The index of $\mathcal{D} = d + \delta$ can be defined as

$$\mathrm{Ind}\,\mathcal{D} := \dim\ker\mathcal{D}^+ - \dim\ker\mathcal{D}^-$$

where

$$\mathcal{D}^+ : \Omega^{\mathrm{even}}(M; \mathbb{R}) \longrightarrow \Omega^{\mathrm{odd}}(M; \mathbb{R})$$

we see that $\omega \in \ker \mathcal{D}$ if and only if ω is harmonic, and from Hodge theorem we know

$$\begin{aligned} \dim \ker \mathcal{D}^+ &= \sum \dim \mathcal{H}^{2k}(M) = \sum \dim H^{2k}(M; \mathbb{R}) \\ \dim \ker \mathcal{D}^- &= \sum \dim \mathcal{H}^{2k-1}(M) = \sum \dim H^{2k-1}(M; \mathbb{R}) \end{aligned}$$

hence we find that

$$\operatorname{Ind} \mathcal{D} = \sum_{k=1}^{n} (-1)^k \dim H^k(M; \mathbb{R}) = \chi(M)$$

which is a topological invariant. ♣

However, for this note, we will use a simpler formula for Spin^c 4-manifolds.

Recall that when $\dim M = 4$, the Hodge $*-$orperator decomposes the 2-forms into the self-dual and anti-self-dual parts, hence there is a same decomposition on all harmonic forms

$$\mathcal{H}^2(M) = \mathcal{H}^2_+(M) \oplus \mathcal{H}^2_-(M)$$

Definition 2.12 (Signature)**.** The signature of a 4-manifold M is defined as

$$\operatorname{sgn}(M) := b^+ - b^- = \dim \mathcal{H}^2_+(M) - \dim \mathcal{H}^2_-(M)$$

Theorem 2.9. *The index of* $\not{D}_A$ *on* $\$_M = \$_+ \oplus \$_-$ *associated to* Spin^c *structure on a 4-manifold is*

$$\begin{aligned} \operatorname{Ind} \not{D}_A &= -\int_M \left(\frac{p_1(M)}{6} - \frac{c_1^2(\mathcal{L})}{2} \right) \\ &= -\frac{\operatorname{sgn}(M)}{8} + \frac{1}{2} \int_M c_1^2(\mathcal{L}) \end{aligned}$$

where $\mathcal{L}$ *is the determinant line bundle of the* Spin^c *structure,* $p_1(M) = c_2(TM \otimes \mathbb{C})$ *is the first Pontryagin class.*

Here is an application of the index theorem, I refer to [14] for more interesting applications.

Theorem 2.10 (Linchnerowicz)**.** *If a compact oriented 4-manifold* M *is spin, then it admits a Riemannian metric* g *with positive scalar curvature* $s_g(x) > 0$ *only if its signature is zero.*

Proof. Consider the Spin^c structure which is twisted by a trivial line bundle L from the original spin structure. Then by Example 2.1, we know that

$$c_1(\mathcal{L}) = c_1(L) = 0$$

where $\$_M = \$_+ \oplus \$_-$ is the spinor bundle associated to the spin structure. So by Atiyah-Singer index theorem, we have

$$-\frac{\operatorname{sgn}(M)}{8} = \operatorname{Ind} \not{D}_A$$

Now, we pick A to be a trivial connection on $\mathcal{L} = L$. If $\psi \in \Gamma(\$_+ \otimes L)$ satisfies $\not{D}_A^+ \psi = 0$, we have by Weitzenböck formula (24)

$$0 = \left(\not{D}_A^- \not{D}_A^+ \psi, \psi \right)_{L^2} = (\nabla_A^* \nabla_A \psi, \psi)_{L^2} + \left(\frac{s_g}{4} \psi, \psi \right)_{L^2} + \left(\frac{F_A^+ \cdot \psi}{2}, \psi \right)_{L^2}$$
$$= \|\nabla_A \psi\|_{L^2}^2 + \frac{1}{4} \int_M s_g(x) |\psi(x)|^2 \mathrm{vol}_M$$

thus if $s_g(x) > 0$ everywhere, there could only be $\psi = 0$, therefore $\mathrm{Ind}\, \not{D}_A = 0$, that yields $\mathrm{sgn}(M) = 0$. ♣

By the same argument, we can prove a much stronger result:

Theorem 2.11. *A compact $2n-$dimensional spin manifold M admits a Riemannian metric whose scalar curvature is positive everywhere only if the $\hat{A}$-genus of M is zero:*

$$g_{\hat{A}}(M) = \int_M \hat{A}(M) = 0$$

Part II: Seiberg-Witten Gauge Theory

3 Seiberg-Witten Equations

3.1 Definitions and Seiberg-Witten Maps

Now we can formulate what is a Seiberg-Witten equation. From this section (M, g) will be a compact oriented Riemannian 4-manifold, and we only pick a Spin^c structure on it, the spinor will be denoted by $\$_M = \$_+ \oplus \$_-$ and $\mathcal{L} = \det \$_+ = \det \$_-$ be its determinant line bundle.

We shall first define a quadratic map μ on $\$_+$.

Definition 3.1. For a $\psi \in \Gamma(\$_+)$, define

$$\mu(\psi) = \psi \otimes \psi^* - \frac{|\psi|^2}{2}\mathrm{Id} \in \$_+ \otimes \$_+^* \cong \mathrm{End}(\$_+)$$

where ψ is the dual spinor of ψ in sense of the Hermitian metric. ♣

Remark 3.1. Since $\$_+$ has complex rank 2, we can locally write $\psi = (\psi_1, \psi_2)$, then μ computes as

$$\mu : \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \longrightarrow \frac{1}{2}\begin{pmatrix} |\psi_1|^2 - |\psi_2|^2 & 2\psi_1\bar{\psi}_2 \\ 2\bar{\psi}_1\psi_2 & |\psi_2|^2 - |\psi_1|^2 \end{pmatrix} \in \sqrt{-1}\mathfrak{su}(\$_+)$$

hence by remark 2.4 (iv), we regard $\mu(\psi)$ as a pure imaginary self-dual 2-form.

Definition 3.2. The Seiberg-Witten equation is an equation of $(\psi, A) \in \Gamma(\$_+) \times \mathcal{A}(\mathcal{L})$:

$$\begin{cases} \not{D}_A^+\psi = 0 \\ F_A^+ = \mu(\psi) \end{cases} \tag{25}$$

Example 3.1. Let $M = \mathbb{R}^4$, let's try to write down what a Seiberg-Witten equation looks likes (or leave as an exercise). Well, in this case a spinor field with the positive chirality ψ can be viewed as a quaternionic valued function on $\mathbb{R}^4$, and $A = d + a$ a connection on the line bundle $\mathcal{L} = \mathbb{R}^4 \times \mathbb{C}$ is simply $\sum_{i=0}^3 a_i(x)dx_i$, by example 2.4 (18), we see that those dx_i's acts on ψ by multiplying with $1, i, j, k$ respectively, thus ($\sqrt{-1}$ action in $\mathbb{H}$ can be take as multiplying with i)

$$\not{D}_A^+\psi = \not{D}^+\psi + \frac{i}{2}a \cdot \psi$$

by example 2.5, we know that $\not{D}^+$ is just the Cauchy-Riemann operator, $a \cdot \psi$ contains only the products of components of ψ and a.

The second equation involves $F_A^+ = da$, which contains the 1st order linear partial differentials of a_i's, and $\mu(\psi)$ is a 0th order non-linear term.

Therefore, the Seiberg-Witten equation has the form

$$\text{1st order linear derivatives} + \text{0th order non-linear terms} = 0$$

which is a not extremely non-linear PDE. ♣

We can apply Weitzenböck formula to obtain a very coarse property of the solutions of (25) .

Theorem 3.1. *If the Riemannian 4-manifold (M, g) has positive scalar curvature, then every solution of the Seiberg-Witten equation (25) has the form $(0, A)$.*

Proof. Assume (ψ, A) is a solution of (25), then by definition of μ, we have

$$\langle \mu(\psi)\psi, \psi \rangle = \frac{1}{2}|\psi|^4 \tag{26}$$

thus by applying Weitzenböck formula (24), we have

$$\begin{aligned} 0 = \left(\not{D}_A^- \not{D}_A^+ \psi, \psi \right)_{L^2} &= \|\nabla_A \psi\|_{L^2}^2 + \frac{1}{2}(\mu(\psi)\psi, \psi)_{L^2} + \frac{1}{4}\int_M s_g |\psi|^2 \mathrm{vol}_M \\ &= \|\nabla_A \psi\|_{L^2}^2 + \frac{1}{4}\int_M |\psi|^2 \left(s_g + |\psi|^2 \right) \mathrm{vol}_M \end{aligned}$$

hence if $s_g > 0$ for all $x \in M$, then there could only be $\psi = 0$. ♣

It is convenient to define the Seiberg-Witten map.

Definition 3.3. The Seiberg-Witten map is

$$\begin{aligned} SW : \Gamma(\$_+) \times \mathcal{A}(\mathcal{L}) &\longrightarrow \Gamma(\$_-) \times \Omega_+^2(M; i\mathbb{R}) \\ (\psi, A) &\mapsto \left(\not{D}_A^+ \psi, F_A^+ - \mu(\psi) \right) \end{aligned} \tag{27}$$

We will denoted by $\mathcal{C} := \Gamma(\$_+) \times \mathcal{A}(\mathcal{L})$ which is an infinite dimensional affine space modeled on $\Gamma(\$_+) \times \Omega^1(M; i\mathbb{R})$ and $\mathcal{V} = \Gamma(\$_-) \times \Omega_+^2(M; i\mathbb{R})$ which is an infinite dimensional vector space $\Gamma(\$_-) \times i\mathfrak{su}(\$_+)$. Wee can also see that (ψ, A) solves (25) if and only if $SW(\psi, A) = 0$, thus the space of solutions of the Seiberg-Witten equation is the zero locus $SW^{-1}(0)$.

We can compute the tangent map of the Seiberg-Witten map.

Theorem 3.2. *The tangent map $d_{(\psi,A)}SW$ is*

$$\begin{aligned} d_{(\psi,A)}SW : \Gamma(\$_+) \times \Omega^1(M; i\mathbb{R}) &\longrightarrow \mathcal{V} = \Gamma(\$_-) \times \Omega_+^2(M; i\mathbb{R}) \\ (\Psi, \alpha) &\mapsto \left(\not{D}_A^+ \Psi + \frac{1}{2}\alpha \cdot \psi, (d\alpha)^+ - 2\mu(\psi, \Psi) \right) \end{aligned} \tag{28}$$

where $(d\alpha)^+$ stands for taking self-dual part.

Proof. Just a straightforward computation, since

$$\gamma(t) = (t\Psi + \psi, A + t\alpha) : I \longrightarrow \mathcal{C} = \Gamma(\not{S}_+) \times \mathcal{A}(\mathcal{L})$$

is a parameterized curve with $\gamma(0) = (\psi, A)$ and $\gamma'(0) = (\Psi, \alpha)$. Then the computation runs as

$$d_{(\psi,A)}SW(\Psi, A) = \left.\frac{d}{dt}\right|_{t=0} SW(\gamma(t))$$

. ♣

3.2 Gauge Group Actions

The gauge group is $\mathrm{Aut}(\mathcal{L}) \subset \mathrm{End}(\mathcal{L}) = M \times \mathbb{C}$, hence the gauge group can be regarded as

$$\mathcal{G} = C^\infty(M; \mathrm{U}(1))$$

Recall that by remark 2.3, a $g \in \mathcal{G}$ acts on a section s on $\mathcal{L}$ by $g \cdot s = g^2 s$, thus $\mathcal{G}$ acts on $\mathcal{A}(\mathcal{L})$ by

$$A \cdot g = 2g^{-1}dg + A$$

We define the whole gauge group action on $\mathcal{C}$ by

$$(\psi, A) \cdot g := \left(g^{-1}\psi, A \cdot g\right)$$

Lemma 3.1. *The space of solutions of (25) $SW^{-1}(0)$ is gauge invariant under $\mathcal{G}$ action. It is equivalent to say the Seiberg-Witten map is $\mathcal{G}-$equivariant:*

$$\begin{array}{ccc}
\mathcal{C} = \Gamma(\not{S}_+) \times \mathcal{A}(\mathcal{L}) & \xrightarrow{SW} & \mathcal{V} = \Gamma(\not{S}_-) \times \Omega^2_+(M; i\mathbb{R}) \\
\big\downarrow \curvearrowright \mathcal{G} & & \mathcal{G} \curvearrowright \big\downarrow \\
\mathcal{C} & \xrightarrow{\quad SW \quad} & \mathcal{V}
\end{array}$$

where $\mathcal{G}$ acts on $\mathcal{V}$ by multiplying g^{-1} on the first faactor and trivially on the second.

Proof. Let $g \in \mathcal{G}$, we have

$$\begin{aligned}
\not{D}^+_{A\cdot g}(g \cdot \psi) &= \not{D}^+_{A+2g^{-1}dg}\left(g^{-1}\psi\right) \\
&= \not{D}^+_A\left(g^{-1}\psi\right) + g^{-2}dg\psi \\
&= \left(dg^{-1}\right) \cdot \psi + g^{-1}\not{D}^+_A\psi \\
&= g^{-1}\not{D}^+_A\psi = g \cdot \left(\not{D}^+_A\psi\right)
\end{aligned}$$

For $F^+_{A\cdot g}$, we notice that $\mathcal{G}$ is Abelian, thus the gauge transformation doesn't change the curvature hence $F^+_{A\cdot g} = F^+_A$. And since $|g| = 1$, thus by definition of μ we also have $\mu\left(g^{-1}\psi\right) = \mu(\psi)$, thus we obtained

$$SW((\psi, A) \cdot g) = g \cdot SW(\psi, A)$$

which proves the equivariancy. ♣

Now, we can define what is the Seiberg-Witten moduli space.

Definition 3.4. The Seiberg-Witten moduli space is the solutions of (25) modulo the gauge equivalency:

$$\mathcal{M}_{SW} := SW^{-1}(0)/\mathcal{G}$$

Also notice that, the $\mathcal{G}$-action is non-free [19] at (ψ, A) precisely if $\psi = 0$, thus we can define the moduli space for *irreducible solutions*

$$\mathcal{M}_{SW}^{irr} := \left\{(\psi, A) \in SW^{-1}(0) | \psi \neq 0\right\} / \mathcal{G}$$

. ♣

However, one shall never expect $SW^{-1}(0)$ is a smooth submanifold of an infinite dimensional affine space $\mathcal{C}$ since SW may not intersect transversally with 0, hence one shall never expect $\mathcal{M}_{SW}$ or $\mathcal{M}_{SW}^{irr}$ is smooth. In next section, we will discuss how deal with the "transversality" problem.

[19] A G action at $x \in M$ is called *free* if the stabilizer $G_x \subset G$ is trivial. It means the G-orbit at a free point just looks likes G it self. Hence consequently, if we can take an open subset U of a manifold M such that each point in U is a free point of the G-action, then the union of all orbits through U looks likes $U \times G$, it defines a coordinate chart in the quotient space M/G. Therefore, if G-action is free everywhere, the orbit space M/G is a manifold, and the quotient map $\pi : M \longrightarrow M/G$ provides that M is a principal G-bundle over M/G.

In general, that U such that $\pi^{-1}(U) = U \times G$ is called a **slice**.

4 Seiberg-Witten Moduli Spaces

This section aims to introduce three main properties of Seiberg-Witten moduli spaces. The first remarkable property is that $\mathcal{M}_{SW}$ is compact in C^∞-topology (i.e. the space for non-equivalent *smooth* solutions is compact). The second property is that SW can be *perturbed* by a very small parameter η so that SW_η intersects transversally with 0. The third property is that we can give an orientation on the moduli spaces.

4.1 Compactness of $\mathcal{M}_{SW}$

We wish to prove $\mathcal{M}_{SW}$ is a compact moduli space in sense of $C^\infty-$topology. Our strategy will be as follows:

- $C^\infty-$topology is a bit strong, we will first replace it by a slightly weaker topology, induced by the *Sobolev norm* $W^{k,2}$. Here we shall take k sufficiently large ($k = 5$ for instance) so that some definitions work. The moduli space of $W^{5,2}-$solutions will be denoted by $\mathcal{M}^{5,2}$
- Then, we will show all solutions of (25) are (up to a gauge equivalence) actually bounded in $W^{6,2}$-norm. Actually, they are (up to a gauge equivalence) bounded in any $W^{k,2}$-norm, for $k \geq 0$.
- Apply Sobolev embedding theorem ($W^{k+1,2} \hookrightarrow W^{k,2}$ is compact embedding) we know that $\mathcal{M}^{5,2}$ is compact. Since for a sequence $\{(\psi_n, A_n)\}$ in $\mathcal{C}^{5,2} \cap SW^{-1}(0)$, we know that it is gauge equivalent to a sequence which is bounded in $W^{6,2}-$norm, hence by Sobolev embedding (cf. Theorem 4.1), there is a subsequence which is convergent in $W^{5,2}-$norm, then by definition (of sequential compactness) the moduli space $\mathcal{M}^{5,2}$ is compact.
- By upgrading k to any $k \geq 0$, we can show that $\mathcal{M}_{SW}$ is compact under C^∞-topology.

With this strategy in mind, let's start from the Sobolev completions.

4.1.1 Sobolev Completions

Recall that the space $\Gamma(\$_+)$ admits an Hermitian L^2-norm

$$\|\psi\|_{L^2}^2 := \int_M \langle \psi, \psi \rangle_x \mathrm{vol}_M = \int_M |\psi(x)|^2 \mathrm{vol}_M$$

we can define its Sobolev norm by

$$\|\psi\|_{W^{k,2}}^2 = \sum_{i=0}^{k} \left\|\nabla_A^i \psi\right\|_{L^2}^2 = \int_M \sum_{i=0}^{k} \left|\nabla_A^i \psi\right|^2 \mathrm{vol}_M$$

where ∇_A is the spin connection, it acts on ψ can be viewed as acting on $(\psi, 0) \in \Gamma(\$_M)$.

The Sobolev completion of $\Gamma(\$_+)$ under $W^{k,2}$-norm is denoted by

$$W^{k,2}(\$_+) := \overline{(\Gamma(\$_+), \|\cdot\|_{W^{k,2}})}$$

It is a Banach space.

Similarly, by Hodge theory, we know there is an L^2-norm on $\Omega^p(M;\mathbb{R})$, defined by

$$\|\omega\|_{L^2}^2 = \int_M \omega \wedge *\omega$$

Its Sobolev $W^{k,2}$-norm is defined by

$$\|\omega\|_{W^{k,2}}^2 := \sum_{i=0}^{k} \left\|D^i\omega\right\|_{L^2}^2$$

where D is the induced Levi-Civita connection on $\bigwedge^p T^*M$. The sobolev completion of $\Omega^p(M;\mathbb{R})$ will be denoted by:

$$W^{k,2}\left(\bigwedge^k M \otimes \mathbb{R}\right) := \overline{(\Omega^p(M;\mathbb{R}), \|\cdot\|_{W^{k,2}})}$$

Notice that after picking a frame point A_0, the space of connections $\mathcal{A}(\mathcal{L})$ is an affine space modeled on $\Omega^1(M;i\mathbb{R})$:

$$\mathcal{A}(\mathcal{L}) = A_0 + \Omega^1(M;i\mathbb{R})$$

we can also define the $W^{k,2}-$Sobolev completion on $\mathcal{A}(\mathcal{L})$, denoted by $\mathcal{A}^{k,2}(\mathcal{L})$.

Let

$$\mathcal{C}^{k,2} = \left\{(\psi, A) \in \mathcal{C} | (\psi, A) \in W^{k,2}(\$_+) \times \mathcal{A}^{k,2}(\mathcal{L})\right\}$$

be the Banach affine space, and

$$\mathcal{K}^{k,2} = \left\{(\psi, A) \in \mathcal{C}^{k,2} | SW(\psi, A) = 0\right\}$$

It is also possible to give a Sobolev completion on $\mathcal{G} = C^\infty(M;\mathrm{U}(1))$, denoted by $\mathcal{G}^{k,2}$, however, from Sobolev multiplication theorem 4.1, we know that it is a Banach Lie group whenever $k \geq 3$.

Here are the fundamental theorems of Sobolev spaces:

Theorem 4.1. *Let M be an n-dimensional manifold, $E \longrightarrow M$ an Euclidean vector bundle. $W^{k,p}(E)$ is the $W^{k,p}-$Sobolev completion of $\Gamma(E)$, then*

(1) If $s \in W^{k,p}(E)$, then $s \in W^{\ell,q}(E)$, where

$$k - \ell \geq n\left(\frac{1}{p} - \frac{1}{q}\right) \geq 0$$

and there exists a constant $C > 0$ such that

$$\|s\|_{W^{\ell,q}} \leq C\|s\|_{W^{k,p}}$$

It is equivalent to say there is a continuous embedding

$$j : W^{k,p}(E) \hookrightarrow W^{\ell,q}(E)$$

(2) *(Sobolev Embedding Theorem) If*

$$k - \ell \geq n\left(\frac{1}{p} - \frac{1}{q}\right) > 0$$

then the embedding j is compact.

(3) *If $k - \frac{n}{p} > r$, then there is a continuous embedding*

$$W^{k,p}(E) \hookrightarrow C^r(E)$$

If $s \in W^{k,p}$ for some fixed p but for all $k \geq 0$, then s is smooth.

(4) *(Sobolev Multiplication Theorem)*

- *If $kp > n$, then the Banach space $W^{k,p}(M;\mathbb{R})$ is a Banach algebra.*
- *If $kp < n$, then we have a bounded map*

$$W^{k_1,p_1} \otimes W^{k_2,p_2} \longrightarrow W^{k,p}$$

whenever

$$k_1 - \frac{n}{p_1} + k_2 - \frac{n}{p_2} \geq k - \frac{n}{p}$$

Here are the main analytic properties of the operator $\not{D}_A$ and F_A^+.

Theorem 4.2. *Let $A_0 \in \mathcal{A}(\mathcal{L})$ be fixed.*

(1) *(Elliptic Estimation) The Dirac operator $\not{D}_{A_0}$ is a 1st order elliptic operator, it satisfies the* ***elliptic estimation***

$$\|\psi\|_{W^{k+1,2}} \leq C\left(\|\not{D}_{A_0}\psi\|_{W^{k,2}} + \|\psi\|_{L^2}\right)$$

(2) *(Gauge Fixing Lemma) For any $A \in \mathcal{A}(\mathcal{L})$, there exists a gauge transformation $g \in \mathcal{G}$ with $A \cdot g = A_0 + \alpha$ such that*

$$\begin{cases} \delta\alpha = 0 \\ \|\alpha\|_{W^{k,2}} \leq C_1\|F_A^+\|_{W^{k-1,2}} + C_2 \end{cases}$$

The proof of (1) is due to the fact that all elliptic operators satisfying the elliptic estimations. The proof of the second statement can be found in [2, lemma 5.3.1] (or leave as an exercise).

4.1.2 Proof of the Compactness

Following the proving strategy listed at the beginning, we now focus on the solutions in $\mathcal{K}^{5,2}$. We will show that all $W^{5,2}$-solutions (ψ, A) are uniformly bounded in $W^{k,2}-$norm for all $k \geq 0$.

Lemma 4.1. *There exists a constant $C > 0$ (which only depends on the geometry of (M, g) and not depends on the choice of (ψ, A)) such that for all $(\psi, A) \in \mathcal{C}^{5,2}$*

$$\|\psi\|_{L^p} \leq C$$

Proof. We shall first prove it for C^0-norm:

$$\|\psi\|_{C^0} = \sup_{x \in M} |\psi(x)| \leq C$$

Choose $x_0 \in M$ be a maximal value of $|\psi|^2$ (since M is compact). Let $e_1, ..., e_n$ be the geodesic local frame on $T_{x_0}M$, then by (22), (23), (26) and Weitzenböck formula (24) we have at x_0:

$$\begin{aligned}
0 \leq \triangle|\psi|^2 &= -\sum_{i=1}^{4} \frac{\partial}{\partial e_i} \frac{\partial \langle \psi, \psi \rangle}{\partial e_i} = -2\sum_{i=1}^{4} \frac{\partial \left\langle \nabla_{A_{e_i}} \psi, \psi \right\rangle}{\partial e_i} \\
&= -2\sum_{i=1}^{4} \left(\left\langle \nabla_{A_{e_i}} \nabla_{A_{e_i}} \psi, \psi \right\rangle + \left|\nabla_{A_{e_i}} \psi\right|^2 \right) \\
&= 2 \left\langle \nabla_A^* \nabla_A \psi, \psi \right\rangle - 2 \left|\nabla_A \psi\right|^2 \\
&= 2 \left\langle \not{D}_A^- \not{D}_A^+ \psi, \psi \right\rangle - \frac{s_g(x_0)}{2} |\psi|^2 - \left\langle F_A^+ \cdot \psi, \psi \right\rangle - 2 \left|\nabla_A \psi\right|^2 \\
&= -\frac{s_g(x_0)}{2} |\psi|^2 - \frac{1}{2} |\psi|^4 - 2 \left|\nabla_A \psi\right|^2
\end{aligned}$$

hence we obtained

$$\frac{s_g(x_0) + |\psi(x_0)|^2}{2} |\psi(x_0)|^2 \leq -2 \left|\nabla_A \psi\right|^2 | \leq 0$$

thus

$$|\psi(x_0)|^2 \leq -s_g(x_0) := C$$

Consequently we have

$$\begin{aligned}
\|\psi\|_{L^p}^p := \int_M |\psi|^p \mathrm{vol}_M &\leq \int_M |\psi(x_0)|^p \mathrm{vol}_M \\
&\leq C^p \mathrm{vol}(M)
\end{aligned}$$

which ends the proof. ♣

Lemma 4.2. *For any solution* (ψ, A) *of (25) we have*

$$\|F_A^+\|_{L^2} \leq C, \quad \|F_A^-\|_{L^2} \leq C - 4\pi^2 c_1^2(\mathcal{L})$$

where

$$c_1^2(\mathcal{L}) := \int_M c_1(\mathcal{L}) \smile c_1(\mathcal{L})$$

Proof. The first inequality follows directly from lemma 4.1, as for the second one, notice that by Chern-Weil theory

$$c_1(\mathcal{L}) = \frac{i}{2\pi}\mathrm{Tr}(F_A) = \frac{i}{2\pi}F_A$$

hence

$$\begin{aligned} c_1^2(\mathcal{L}) &= -\frac{1}{4\pi^2}\int_M F_A \wedge F_A \\ &= -\frac{1}{4\pi^2}\int_M \left(F_A^+ + F_A^-\right) \wedge \left(F_A^+ + F_A^-\right) \\ &= -\frac{1}{4\pi^2}\left(\|F_A^+\|_{L^2}^2 - \|F_A^-\|_{L^2}^2\right) \end{aligned}$$

which yields the desired result. ♣

Lemma 4.3. *For each* $k \geq 0$*, there exits a constant* $C_k > 0$ *such that for any solution* $(\psi, A) \in \mathcal{C}^{5,2}$*, there exists a gauge transformation* $g \in \mathcal{G}^{k+1,2}$ *with* $A \cdot g = A_0 + \alpha$ *such that*

$$\|(\psi, \alpha)\|_{W^{k,2}} \leq C_k$$

Proof. We can prove by induction, however we need to establish for some small k until $k = 4$ so that the Sobolev multiplication theorem can be used.

- For $k = 0$, the result follows directly from lemma 4.1.
- For $k = 1$, by gauge fixing lemma and lemma 4.2, there exists a $g \in \mathcal{G}^{2,2}$ with $A \cdot g = A_0 + \alpha$ such that

  $$\|\alpha\|_{W^{1,2}} \leq C_1 \left\|F_A^+\right\|_{L^2} + C_2 \leq C'$$

  thus $\alpha \in W^{1,2}$.

  As for $\|\psi\|_{W^{1,2}}$, note that by Sobolev multiplication $g \cdot \psi$ is still of $W^{0,2} = L^2$, to avoid the abused using of notations, which will still be denoted by ψ, hence by (25), we have

  $$0 = \not{D}_A^+\psi = \not{D}_{A_0}^+\psi + \frac{\alpha \cdot \psi}{2}$$

  hence again by Sobolev multiplication theorem, we have

  $$\not{D}_{A_0}^+\psi = -\frac{\alpha \cdot \psi}{2} \in W^{0,2}(\$_+) = L^2(\$_+)$$

  by elliptic estimation and lemma 4.1, we have

  $$\|\psi\|_{W^{1,2}} \leq C\left(\left\|\not{D}_{A_0}^+\psi\right\|_{L^2} + \|\psi\|_{L^2}\right) \leq C'$$

- For $k = 2$, in order to establish an estimation on $\|\alpha\|_{W^{2,2}}$ we need to study $\left\|F_A^+\right\|_{W^{1,2}}$ by gauge fixing lemma, which entails to study $\left\|DF_A^+\right\|_{L^2}$.

  We still use ∇_A for the induced spin connection on $\operatorname{End}(\not{S}_+) \supset \mathfrak{su}(\not{S}_+)$. By definition, we have

$$\begin{aligned} DF_A^+ &= \nabla_A \mu(\psi) = \nabla_A \left(\psi \otimes \psi^* - \frac{|\psi|^2}{2}\mathrm{Id}\right) \\ &= (\nabla_A \psi) \otimes \psi^* + \psi \otimes \nabla_A \psi^* - \left(\sum_{i=1}^n \langle \nabla_{A_{e_i}} \psi, \psi\rangle\right) \mathrm{Id} \end{aligned}$$

  since $\psi \in W^{1,2}$, by applying triangle inequality we have

$$\left\|DF_A^+\right\|_{L^2} \leq C\|\nabla_A \psi\|_{L^2} \cdot \|\psi\|_{L^2} \leq C'$$

  thus $F_A^+ \in W^{1,2}$, by gauge fixing lemma we can prove $\alpha \in W^{2,2}$ for some $A \cdot g = A_0 + \alpha$.

  Again, by Sobolev multiplication we can show that

$$\not{D}_{A_0}^+ \psi = -\frac{\alpha \cdot \psi}{2} \in W^{1,2}(\not{S}_+)$$

  by applying elliptic estimation again, we have

$$\|\psi\|_{W^{2,2}} \leq C\left(\left\|\not{D}_{A_0}^+\psi\right\|_{W^{1,2}} + \|\psi\|_{L^2}\right) \leq C'$$

- The proof for $k = 3$ is similar, just compute $D^2 F_A^+$ and by applying our previously results.
- Now, we can start the induction, suppose the result holds till to some $k \geq 4$, by Sobolev multiplication we have

$$F_A^+ = \mu(\psi) \in W^{k,2}\left(\bigwedge\nolimits_+^2 T^*M \otimes i\mathbb{R}\right)$$

  hence by gauge fixing lemma, there exits some $g \in \mathcal{G}^{k+2,2}$ provides $A \cdot g = A_0 + \alpha$ such that

$$\|\alpha\|_{W^{k+1,2}} \leq C_1 \left\|F_A^+\right\|_{W^{k,2}} + C_2 \leq C'$$

  The same as before, by applying elliptic estimation, we conclude that $(\psi, \alpha) \in W^{k+1,2}$.

That ends the proof. ♣

As was stated in the beginning strategy, we have

Lemma 4.4. *The moduli space $\mathcal{M}^{5,2} = \mathcal{K}^{5,2}/\mathcal{G}^{6,2}$ is compact.*

As the consequence of theorem 4.1 (3) and lemma 4.3, we obtained

Theorem 4.3. *Every solution* $(\psi, A) \in \mathcal{K}^{5,2}$ *can be gauge transformed to a smooth solution* $(\psi, A) \cdot g \in SW^{-1}(0)$ *for some* $g \in \mathcal{G}^{6,2}$*. Moreover,* $\mathcal{M}_{SW}$ *is compact under* C^∞*-topology.*

Proof. Just show that for each $\mathcal{K}^{5,2}$-solution (ψ, A) there exists a $g \in \mathcal{G}^{6,2}$ such that (ψ, α) is in $W^{k,2}$ for all $k \geq 0$ (be careful that, different from lemma 4.3, where $g \in \mathcal{G}^{k,2}$ depends on k), where $A \cdot g = A_0 + \alpha$.

By lemma 4.3, we know that if (ψ, A) is a $\mathcal{K}^{5,2}$ solution, ψ is in $W^{k,2}$ for all $k \geq 0$, hence ψ is smooth. And by Sobolev multiplication $\|F_A^+\|_{W^{k,2}} = \|\mu(\psi)\|_{W^{k,2}}$ is also bounded below for all $k \geq 0$, hence by lemma 4.2 $F_A = F_A^+ + F_A^-$ is smooth. Notice that if we denote $A \cdot g = A_0 + \alpha$ for some $g \in \mathcal{G}^{6,2}$ we have

$$F_A = F_{A \cdot g} = F_{A_0} + d\alpha$$

don't forget $\delta\alpha = 0$, we have

$$(d + \delta)\alpha = F_A - F_{A_0} = d\alpha$$

thus $d\alpha$ is smooth, hence so is α.

To show $\mathcal{M}_{SW}$ is smooth under C^∞ topology, it suffices to show that for any sequence (ψ_n, A_n) in $\mathcal{M}_{SW}$, there exists a subsequence (ψ_{n_k}, A_{n_k}) converges in $W^{k,2}$-norm for all $k \geq 0$.

Since (ψ_n, A_n) is bounded below in $W^{6,2}-$norm, there exists a subsequence (ψ_{n_i}, A_{n_i}) which converges in $W^{5,2}-$norm after applying a gauge transformation in $\mathcal{G}^{6,2}$. And since (ψ_{n_i}, A_{n_i}) is also in $W^{7,2}$, we can choose a subsequence which converges in $W^{6,2}$-norm (after applying a gauge transformation in $\mathcal{G}^{7,2}$), keep doing this process, and by the choice axiom we can pick the diagonal subsequence which is converge in any $W^{k,2}$-norm. ♣

4.2 Smoothness

As was stated at the beginning of this section, if we can show that $SW^{-1}(0)$ is smooth, then at least $\mathcal{M}_{SW}^{irr}$ is a smooth manifold. However, although SW may not intersect transversally with 0, we can perturb it by a small constant $\eta \in W^{k,2}\left(\bigwedge_+^2 T^*M \otimes i\mathbb{R}\right)$ so that good thing happens. In order to do so, we need to establish an $\infty-$dimensional Thom-Smale theorem.

4.2.1 Fredholm Theory

Let's start from the general Fredholm theory. Let X, Y be two $\infty-$dimensional Banach manifolds, $F : X \longrightarrow Y$ is a smooth map

Definition 4.1 (Fredholm map)**.** Let $y = F(x) \in Y$, the smooth map F is said to be *Fredholm* if $(dF)_x; T_xX \longrightarrow T_yY$ is a Freholm map for all x, that is

(i) $\dim\ker(dF)_x < \infty$

(ii) $\dim\left(Y/\mathrm{Im}(dF)_x\right) < \infty$, that is its cokernel is of finite dimensional.

(iii) $\mathrm{Im}(dF)_x$ is closed [20] in Y.

If X is connected, there is a well-defined *index* of a Fredholm map, which is defined by

$$\mathrm{Ind}\, F := \dim\ker(dF)_x - \dim\mathrm{coker}\,(dF)_x$$

. ♣

A very important class of Fredholm maps is:

Theorem 4.4. *Every elliptic operator of order ℓ*

$$D : W^{k,2}(M;E) \longrightarrow W^{k+\ell,2}(M;E)$$

is a Fredholm operator. Hence in particular, the Dirac operator $\not{D}$ or $\not{D}_A$ are Fredholm, and the Fredholm index equals to the index as elliptic operators.

In what follows, we always assume X, Y are connected.

The Fredholm maps in $\infty-$dimensional differential topology have a lot of good properties as the smooth maps in finite-dimensional topology. For example, the Sard theorem.

Theorem 4.5 (Sard-Smale-Kuranishi)**.** *Assume $F : X \longrightarrow Y$ is a Fredholm map.*

(i) If $0 \in Y$ is a regular value, i.e. $(dF)_x$ is surjective for all x such that $F(x) = 0$, then $F^{-1}(0) \subset X$ is a smooth submanifold of dimension $\mathrm{Ind}F$.

(ii) If X, Y are both paracompact, then the set of regular values of F is a subset in Y of second category, in particular dense.

If the Fredholm map F has index 0 and is also **proper**, then for a regular value 0, $F^{-1}(0)$ is compact 0-dimensional manifold, i.e. finite points. It allows us to define what is a $\mathbb{Z}_2-$mapping degree.

Lemma 4.5. *(a) For any two regular values $y_1, y_2 \in Y$ of F, we have the modulo 2 cardinality*

$$\#F^{-1}(y_2) = \#F^{-1}(y_2) \mod 2$$

*this $\mathbb{Z}_2-$integer is called the $\mathbb{Z}_2-$**degree** of F, denoted by* $\deg_2 F$.

(b) For two homotopic Fredholm maps F_1, F_2, i.e. they were joined by a Fredholm path in $C^\infty(X;Y)$, then

$$\deg_2 F_1 = \deg_2 F_2$$

A more generalized notion of regular values is the transversality.

[20] In fact, (i) & (ii) implies (iii).

Definition 4.2 (Transversality). Let $Z \subset Y$ be a smooth finite dimensional submanifold, a map $F : X \longrightarrow Y$ is said to be *transverse* to Z if for any $z \in Z$ and $x \in F^{-1}(z)$, we have

$$\mathrm{Im}(dF)_x + T_z Z = T_z Y$$

denoted by $F \pitchfork Z$. ♣

Of course, if every $z \in Z$ is a regular value of F then $F \pitchfork Z$, but the reverse side is not necessarily true. Like in the finite dimensional differential topology, we have

Theorem 4.6. *Suppose $F : X \longrightarrow Y$ is Fredholm, $Z \subset Y$ is a finite dimensional submanifold and $F \pitchfork Z$, then $F^{-1}(Z)$ is a smooth submanifold in X with the dimension*

$$\dim F^{-1}(Z) = \mathrm{Ind}\, F + \dim Z$$

Next important question is to generalize the Thom-Smale theorem: For a given submanifold $Z \subset Y$, are the Fredholm maps such that $F \pitchfork Z$ *generic* enough?

To answer this question, we consider a family of Fredholm maps $\{F_w\}$ parameterized in a connected Banach space W, i.e. a Fredholm map

$$\mathcal{F} : X \times W \longrightarrow Y$$

such that for each parameter $w \in W$, $\mathcal{F}(x, w) := F_w$ is Fredholm. An analogously result of Thom-Smale theorem reads:

Theorem 4.7. *Let $y \in Y$ be a regular value of $\mathcal{F}$, then there is subset $W_0 \subset W$ of second category (hence dense) such that y is a regular value of F_{w_0} for each $w_0 \in W_0$.*

That is to say, if for a Fredholm map $F = F_{w_0}$, the value $y \in Y$ may fail to be a regular value for F, but we can always perturb F slightly in the parameter space W so that the perturbed F_w is very *close* to F and regular at y.

Now, if we assume in addition that F_w is proper for each w, and $\mathrm{Ind}\, F = 0$, we can compute the $\mathbb{Z}_2$-mapping degree of $F = F_{w_0}$ at any value y by

$$\deg_2 F := \deg_2 F_w = \# F_w^{-1}(y) \mod 2$$

where the F_w was chose to be perturbed to be regular at y. But there is a natural question: what if one chooses a different perturbation $F_{w'}$?

Lemma 4.6. *Let $\mathcal{F}$ and $F_{w_0} = F$ be defined as above. If W is connected, for any two $w_1, w_2 \in W$ such that $y \in Y$ is a regular of F_{w_i}, $i = 1, 2$, then*

$$\deg_2 F_{w_1} = \deg_2 F_{w_2}$$

Proof. Since W is connected, we can choose a path $\gamma : [0, 1] \longrightarrow W$ joining them, i.e. $\gamma(0) = w_1$, $\gamma(1) = w_2$. Then for each $\gamma(t)$, $\mathcal{F}|_{X \times \{\gamma(t)\}} = \mathcal{F}(x, \gamma(t))$ defines a homotopy between F_{w_1} and F_{w_2}, then by homotopic invariance of $\mathbb{Z}_2$-degree (cf. Lemma 4.5 (b)), the desired result yields. ♣

Next, we wish to give an orientation on $F^{-1}(y)$ (y is regular) so that we can define a $\mathbb{Z}-$valued degree. Recall that an n-dimensional manifold M is orientated if and only if the determinant line bundle $\bigwedge^n TM$ is trivial. Hence for an index d Fredholm map F, we can also study the $\bigwedge^d TF^{-1}(y)$ to give an orientation of $F^{-1}(y)$.

To do this, notice that for $x \in F^{-1}(y)$, the tangent space is actually

$$T_x F^{-1}(y) = \ker(dF)_x$$

However, in practice, it's convenient for us to define:

Definition 4.3. The *determinant line bundle determined by a Fredholm map* $F : X \longrightarrow Y$ is

$$\det F := \coprod_{x \in X} \left(\bigwedge^{\text{top}} \ker(dF)_x \otimes \bigwedge^{\text{top}} \operatorname{coker}(dF)_x \right) \longrightarrow X$$

Lemma 4.7. *The line bundle* $\det F$ *defined in definition 4.3 is indeed a locally trivial real line bundle over* X.

A proof can be found in [15, Appendix A.2.2].

Hence we see that if $\det F$ is a trivial line bundle, then for any regular value y, $F^{-1}(y)$ is an oriented submanifold. In particular, if X, Y are Banach *spaces* and F is a linear Fredholm map, then $\det F$ is obviously trivial.

Now, for the perturbed case $\mathcal{F} : X \times W \longrightarrow Y$, where W is a connected Banach manifold, and let $F_w = \mathcal{F}(x, w)$. Notice that for two parameters $w_1, w_2 \in W$, there exists a path $\gamma(t)$ joining them, thus $\mathcal{F}(x, \gamma(t))$ defines a homotopy between F_{w_1} and F_{w_2}. Moreover, we have

Lemma 4.8. $\det F_{w_1}$ *is trivial if and only if* $\det F_{w_2}$ *is trivial.*

With this in mind, we can define what is the $\mathbb{Z}-$valued mapping degree of a Fredholm map $F : X \longrightarrow W$. Recall that if F is proper and $\operatorname{Ind} F = 0$ then $F^{-1}(y)$ is just a finite set for y regular. If $\det F$ is in additional trivial, then an orientation in $F^{-1}(y)$ means there is a well-defined sign ± 1 on each element in $F^{-1}(y)$, we define $\deg F$ to be the *signed counting* of its cardinality.

If y is not a regular value, then we can choose a perturbation $\mathcal{F} : X \times W \longrightarrow Y$ with $F = F_{w_0}$ for some $w_0 \in W$, then by Thom-Smale theorem 4.7, we can choose a generic w so that y is a regular value of F_w. Now, if we assume in additional that such a perturbation was chose such that $\det F_w$ is trivial for all $w \in W$, then $F_w^{-1}(y)$ is an oriented 0-dimensional compact manifold, i.e. finitely many points with a sign, we define

$$\deg F := \deg F_w$$

By lemma 4.8 we know that the result doesn't depend on the choice of the perturbation F_w.

Like lemma 4.5, we have

Lemma 4.9. *The* $\mathbb{Z}$*-valued mapping degree* $\deg F$ *is independent of the choice of regular value* $y \in Y$ *and it is homotopic invariant.*

4.2.2 Transversality and Perturbed Smoothness

One cannot expect neither 0 is a regular valued of SW nor SW is Fredholm[21]. Hence we need a bit more work to apply the Fredholm theory.

Indeed, there is a classical method to deal with this problem, let me introduce it here.

Let $\mu : \mathcal{C} \longrightarrow \mathcal{V}$ be a smooth map between *Hilbert* manifolds (where $\mathcal{C}$ is affine and $\mathcal{V}$ is linear in our situation) which may be not Fredholm, moreover, $\mathcal{C}$ and $\mathcal{V}$ both endows with a $\mathcal{G}$-action and μ is assumed to be $\mathcal{G}-$equivariant. Let $0 \in \mathcal{V}$ be a fixed point of the $\mathcal{G}-$action, but it might be not a regular value of μ. We wish to study the smoothness of the quotient space $\mu^{-1}(0)/\mathcal{G} := \mathcal{M}$, the so-called *moduli space*. However, there are several problems which may cause the bad behaviors of $\mathcal{M}$.

- $\mu^{-1}(0)$ may not be smooth, since 0 may not be a regular value of μ.
- Even if $\mu^{-1}(0)$ is smooth, $\mathcal{G}$-action on $\mu^{-1}(0)$ may not be free [22].
- Even if $\mathcal{G}$ acts freely on $\mu^{-1}(0)$, $\mathcal{M}$ still may fail to be smooth, since $\mu^{-1}(0)$ may of infinite dimensional (μ may be not Fredholm).
- One may very hard to find a slice S_x at each point $x \in \mu^{-1}(0)$, so that those slices provide the coordinate charts on $\mathcal{M}$.

Then, we aim to discuss a *parameterized* smoothness with the following objectives:

(i) Introduce a parameter space W and define a perturbed map $\mathcal{F} : \mathcal{C} \times W \longrightarrow \mathcal{V}$, such that $\mu = \mathcal{F}(\cdot, \eta_0)$ for some $\eta_0 \in W$, and 0 is a regular value of $\mathcal{F}$ (this $\mathcal{F}$ is not necessarily Fredholm either).

(ii) For generic $\eta \in W$, $\mu_\eta = \mathcal{F}(\cdot, \eta)$ is regular at 0 so that $\mu_\eta^{-1}(0)$ is smooth.

(iii) For each $x \in \mu_\eta^{-1}(0)$ there is a slice S_x so that one can deduce $\mathcal{M}_\eta := \mu_\eta^{-1}(0)/\mathcal{G}$ is smooth.

(iv) For different η_1 and η_2, the moduli spaces $\mathcal{M}_{\eta_1}$ and $\mathcal{M}_{\eta_2}$ are cobordant equivalent.

The most difficult part is obtaining the "generic property" in (ii), our method in here is to use an *elliptic complex*. Here are the detailed approaches.

(1) For each $x \in \mu_\eta^{-1}(0) \subset \mathcal{C}$, consider the following *deformation complex*:

$$0 \longrightarrow \mathrm{Lie}(\mathcal{G}) \xrightarrow{R_x} T_x\mathcal{C} \xrightarrow{(d\mu_\eta)_x} \mathcal{V} \longrightarrow 0$$

where $R_x(\xi) := \underline{\xi}(x)$ means evaluation at x of the fundamental vector field determined by $\xi \in \mathrm{Lie}(\mathcal{G})$. This is indeed a complex due to the equivariancy and 0 is a fixed point:

$$(d\mu_\eta)_x\, \underline{\xi}(x) = \underline{\xi}\left(\mu_\eta(x)\right) = 0$$

[21] Actually, it is a Fredholm part plus a compact part.

[22] Recall that if M is compact and of finite dimensional, then for every *free* compact Lie group G-action, M/G is a smooth manifold. Indeed, for any orbit $G \cdot x \in M/G$, by *Kozul's slice theorem* [16], there exists a $G-$equivariant tubular neighborhood V_x of the orbit $G \cdot x$, such that V_x is equivariantly diffeomorphic to a tubular neighborhood of the zero section of the normal bundle of $G \cdot x$, thus such a V_x forms a slice in M, hence M/G is a smooth manifold (with coordinates provided by these V_x).

(2) Notice that the $\ker (d\mu_\eta)_x$ is precisely $T_x\mu_\eta^{-1}(0)$, hence the 1st cohomology group of this complex is

$$H^1 := T_x\mathcal{M}_\eta$$

(3) Since $\mathcal{C}, \mathcal{V}$ are Hilbert manifolds, we can consider the following map:

$$D_x^\eta := \left(R_x^*, (d\mu_\eta)_x\right) : T_x\mathcal{C} \longrightarrow \mathrm{Lie}(\mathcal{G}) \oplus \mathcal{V}$$

where $R_x^* : T_x\mathcal{C} \longrightarrow \mathrm{Lie}(\mathcal{G})$ is the dual map of R_x.

We shall notice that $\ker R_x^* = (\mathrm{Im}\, R_x)^\perp$, hence

$$\begin{aligned} \ker D_x^\eta &= \ker R_x^* \cap \ker((d\mu_\eta)_x) \\ &= (\mathrm{Im}\, R_x)^\perp \cap T_x\mu_\eta^{-1}(0) \\ &\cong H^1 \cong T_x\mathcal{M}_\eta \end{aligned} \tag{29}$$

(4) Also notice that $\mathrm{Im}\, R_x^* = (\ker R_x)^\perp$, hence if $\mathcal{G}$ acts freely on $\mu_\eta^{-1}(0)$ then R_x^* is also surjective. Under this assumption, we see that 0 is a regular value of μ_η if and only if D_x^η is also surjective.

(5) Thus if we can show in additional that D_x^η is a Fredholm operator, then for generic $\eta \in W$, we have $\mu_\eta^{-1}(0)$ is smooth.

To sum up, we obtain the following:

Theorem 4.8. *If the perturbation $\mathcal{F} : \mathcal{C} \times W \longrightarrow \mathcal{V}$ of μ satisfying the following:*

(a) 0 is a regular value of $\mathcal{F}$.

(b) $\mathcal{G}$ acts freely on $\mu_\eta^{-1}(0)$ for each η.

(c) For each η and $x \in \mu_\eta^{-1}(0)$, there exists a $\mathcal{G}$-slice S_x.

(d) D_x^η is a Fredholm operator of index d for each η.

Then for generic $\eta \in W$, the moduli space $\mathcal{M}_\eta$ is a smooth manifold of dimension d, and if W is connected, two different perturbed moduli spaces are cobordant.

Now, let's deal with the Seiberg-Witten map. For simplicity, we shall only consider the irreducible solutions $\mathcal{C}_{irr}^{5,2}$ so that theorem 4.8 (b) will hold.

Definition 4.4. The perturbed Seiberg-Witten map $\mathcal{SW}$ is

$$\mathcal{C}_{irr}^{5,2} \times W^{4,2}\left(\bigwedge\nolimits_+^2 T^*M \otimes i\mathbb{R}\right) \longrightarrow \mathcal{V}^{4,2} = W^{4,2}(\$_-) \oplus W^{4,2}\left(\bigwedge\nolimits_+^2 T^*M \otimes i\mathbb{R}\right)$$
$$(\psi, A, \eta) \mapsto \left(\not{D}_A^+\psi, F_A^+ - \mu(\psi) - \eta\right)$$

Lemma 4.10. 0 *is a regular value of $\mathcal{SW}$, hence it satisfies Theorem 4.8 (a).*

Proof. Notice that

$$\mathrm{pr}_2 \frac{\partial \mathcal{SW}}{\partial \eta} = -\mathrm{Id} : W^{4,2}\left(\bigwedge\nolimits_+^2 T^*M \otimes i\mathbb{R}\right) \longrightarrow W^{4,2}\left(\bigwedge\nolimits_+^2 T^*M \otimes i\mathbb{R}\right)$$

is surjective, to show $d_{(\psi,A,\eta)}\mathcal{SW}$ is surjective for $\mathcal{SW}(\psi, A, \eta) = 0$ it suffices to show

$$T := \mathrm{pr}_1 d_{(\psi,A)} SW_\eta : T_{(\psi,A)}\mathcal{C}^{5,2}_{irr} \longrightarrow W^{4,2}(\$_-)$$

is surjective for each η. It suffices to show $(\mathrm{Im}\, T)^\perp$ is zero.

By (28), we know that

$$T(\Psi, \alpha) = \not{D}_A^+ \Psi + \frac{\alpha \cdot \psi}{2}$$

Now, if $\varphi \in (\mathrm{Im}\, T)^\perp$ which is non-zero, then we have

$$\left\langle \varphi, \not{D}^+ \Psi + \frac{\alpha \cdot \psi}{2} \right\rangle = 0 \tag{30}$$

for all $(\Psi, \alpha) \in T_{(\psi,A)}\mathcal{C}^{5,2}_{irr}$. In particular, let $\alpha = 0$, we have

$$0 = \left\langle \not{D}_A^+ \Psi, \varphi \right\rangle = \left\langle \Psi, \not{D}_A^- \varphi \right\rangle \Longrightarrow \not{D}_A^- \varphi = 0 \tag{31}$$

Similarly, if we let $\Psi = 0$ in (30), then we have for all $\alpha \in W^{5,2}(T^*M \otimes i\mathbb{R})$:

$$\langle \alpha \cdot \psi, \varphi \rangle = 0 \tag{32}$$

Since ψ satisfies $\not{D}_A^+ \psi = 0$, by the rigidity property [23], there exists an open subset $U \subset M$ such that ψ does not vanish on U. By the property of spin representation and (32), for any α supports on U, one always have

$$\langle \alpha \cdot \psi, \varphi \rangle = 0$$

hence φ almost vanishes on U, but by (31), applying again the rigidity property on φ, we conclude that $\varphi = 0$ on M. ♣

Lemma 4.11. *For each $\eta \in W^{4,2}$, and each $(\psi, A) \in SW_\eta^{-1}(0)$, there exits a slice defined by*

$$(\psi, A) + \ker R^*_{(\psi,A)}$$

thus the perturbed map $\mathcal{SW}$ satisfies Theorem 4.8 (c).

[23] The Dirac operators are somehow a generalization of the Cauchy-Riemann operators, so the spinor field ψ with $\not{D}\psi = 0$ should satisfy the rigidity property as the holomorphic functions.

Proof. Note that [24]

$$S_{(\psi,A)} := (\psi, A) + \ker R^*_{(\psi,A)} = (\psi, A) + \left(\operatorname{Im} R_{(\psi,A)}\right)^{\perp}$$

and since $\mathcal{C}^{5,2}_{irr}$ is affine and $\mathcal{G}^{6,2}$ acts freely on it, it suffices to check

$$T_{(\psi,A)} S_{(\psi,A)} \pitchfork \operatorname{Im} R_{(\psi,A)}$$

and it is indeed the case. ♣

Lemma 4.12. *The operator $D^{\eta}_{(\psi,A)} := (R^*_{(\psi,A)}, d_{(\psi,A)} SW_\eta)$ is an elliptic operator for each $\eta \in W^{4,2}$, hence Theorem 4.8 (d) is also satisfied. Moreover, its index is*

$$\operatorname{Ind} D^{\eta}_{(\psi,A)} = \frac{c_1^2(\mathcal{L}) - 2\chi(M) - 3\operatorname{sgn}(M)}{4}$$

where $\mathcal{L}$ is the determinant line bundle of the Spin^c *structure on M.*

Proof. By definition we can check that the deformation complex

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Lie}(\mathcal{G}^{6,2}) & \xrightarrow{R_{(\psi,A)}} & T_{(\psi,A)}\mathcal{C}^{5,2}_{irr} & \xrightarrow{d_{(\psi,A)}SW_\eta} & \mathcal{V}^{4,2} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & C^{6,2}(M; i\mathbb{R}) & & W^{5,2}\left(\mathbb{S}_+ \oplus \left(\bigwedge^2_+ T*M \otimes i\mathbb{R}\right)\right) & & W^{4,2} & & \end{array}$$

is elliptic, hence $D^{\eta}_{(\psi,A)} := \left(R^*_{(\psi,A)}, d_{(\psi,A)} SW_\eta\right)$ is elliptic, in particular Fredholm.

By a straightforward computation, we have

$$R^*_{(\psi,A)}(\Psi, \alpha) = 2\delta\alpha + i\operatorname{Re}\langle \psi, i\Psi \rangle$$

therefore, we can write $D^{\eta}_{(\psi,A)}$ as

$$D^{\eta}_{(\psi,A)} = \begin{pmatrix} \not{D}^+_A & \\ & d^+ + \delta \end{pmatrix} + B := D_0 + B$$

where B is a zeroth order operator. We can define $D_0 + Bt$ a homotopy between $D^{\eta}_{(\psi,A)}$ and D_0, hence we have

$$\operatorname{Ind} D^{\eta}_{(\psi,A)} = \operatorname{Ind} D_0 = \operatorname{Ind} \not{D}^+_A + \operatorname{Ind}(d^+ + \delta)$$

[24] By a direct computation, we find that for any $\xi \in \operatorname{Lie}(\mathcal{G}) \cong C^\infty(M; i\mathbb{R})$:

$$\begin{aligned} R_{(\psi,A)}\xi = \underline{\xi}(\psi, A) &= \left.\frac{d}{dt}\right|_{t=0} \left(e^{-t\xi}\psi, A + 2e^{-t\xi} t e^{t\xi} d\xi\right) \\ &= (-\xi\psi, 2d\xi) \end{aligned}$$

By Atiyah-Singer index theorem 2.9 we have:

$$\text{Ind}\, \not{D}_A^+ = \frac{c_1^2(\mathcal{L}) - \text{sgn}(M)}{8}$$

and by example 2.6 we have

$$\text{Ind}\,(d^+ + \delta) = b_1 - b_0 - b_2^+ = \frac{\chi(M) - \text{sgn}(M)}{2}$$

where b_2^+ is the dimension of self-dual parts of $H^2(M;\mathbb{R})$. ♣

Hence by theorem 4.8 we have:

Theorem 4.9. *For generic $\eta \in W^{4,2}\left(\bigwedge_+^2 T^*M \otimes i\mathbb{R}\right)$, the perturbed moduli space*

$$\mathcal{M}_\eta^{irr} := \left\{(\psi, A) \in \mathcal{C}^{5,2} | \mathcal{SW}(\psi, A, \eta) = 0, \psi \neq 0\right\}$$

is smooth of dimension

$$\dim \mathcal{M}_\eta^{irr} = \frac{c_1^2(\mathcal{L}) - 2\chi(M) - 3\text{sgn}\,(M)}{4}$$

However, when we only talk about the irreducible solutions, $\mathcal{M}_\eta^{irr}$ loses the compactness, to deal with this problem, we need to add a topological restrain:

Theorem 4.10. *If $b_2^+ \geq 1$, then for generic $\eta \in W^{4,2}$, the solutions of $SW_\eta(\psi, A) = 0$ are all irreducible, hence under this assumption, for generic η, the moduli space $\mathcal{M}_\eta$ is smooth and compact.*

Proof. Indeed, pick a reference point $A_0 \in \mathcal{A}(\mathcal{L})$, consider

$$Q = F_{A_0}^+ + \text{Im} d^+ \subset W^{4,2}\left(\bigwedge_+^2 T^*M \otimes i\mathbb{R}\right)$$

This is a subspace of codimension $b_2^+ \geq 1$, in particular Q^c is dense. Now if $\eta \in Q^c$, we see that any solutions of $SW_\eta(\psi, A) = 0$ one never has $\psi = 0$. ♣

4.3 Orientation

Now, we always assume $b_2^+ \geq 1$. As was discussed in § 4.2.1, we hope to establish a determinant line bundle of $T\mathcal{M}_\eta$, by checking the triviality of this bundle we can detect the orientation of $\mathcal{M}_\eta$.

From (29) we know that it suffices to check the det $D_{(\psi,A)}^\eta$. Due to lemma 4.8, we know that it suffices to check the triviality of det D_0. In fact, we have

$$\det D_0 = \det \not{D}_A^+ \otimes \det(d^+ + \delta)$$

since both $\not{D}_A^+$ and $d^+ + \delta$ are linear, we know that det D_0 is trivial. To sum up, we have showed

Theorem 4.11. *If $b_2^+ \geq 1$, then for generic $\eta \in W^{4,2}$, the perturbed Seiberg-Witten moduli space $\mathcal{M}_\eta$ is a smooth compact oriented manifold. For two different perturbations η_1, η_2, the moduli spaces $\mathcal{M}_{\eta_1}$ and $\mathcal{M}_{\eta_2}$ are cobordant equivalent.*

4.4 Seiberg-Witten Invariants

Now, we can define a numerical invariant of 4-manifold M, the celebrated Seiberg-Witten invariant.

Pick $m \in M$, consider the based gauge group:

$$\mathcal{G}_0 := \{g \in \mathcal{G} | g(m) = 1\}$$

then we have a short exact sequence of Lie groups

$$1 \longrightarrow \mathcal{G}_0 \hookrightarrow \mathcal{G} \xrightarrow{\text{ev}_m} \text{U}(1) \longrightarrow 1$$

where ev_m means evaluation at m.

The quotient space

$$\hat{\mathcal{M}}_\eta := SW_\eta^{-1}(0)/\mathcal{G}_0$$

is called the *framed moduli space*. It equips with a free $\text{U}(1)-$action, and $\hat{\mathcal{M}}_\eta/\text{U}(1) \cong \mathcal{M}_\eta$. Which is to say, $\hat{\mathcal{M}}_\eta$ is a $\text{U}(1)-$bundle over $\mathcal{M}_\eta$, let's denote the 1st Chern class of this bundle by $\omega \in H^2(\mathcal{M}_\eta, \mathbb{Z})$.

Let $d = \dim \mathcal{M}_\eta$ and $\mathcal{S}(M)$ the affine space of Spin^c structures on M, then by theorem 4.11, there is a well-defined numerical invariant:

Definition 4.5 (Seiberg-Witten Invariant)**.** The Seiberg-Witten invariant sw of M is a function $sw : \mathcal{S}(M) \longrightarrow \mathbb{Z}$, defined by

$$sw(\sigma) := \begin{cases} \displaystyle\int_{\mathcal{M}_\eta} \omega^{\frac{d}{2}} & \text{if } d \text{ is even} \\ 0 & \text{if } d \text{ is odd} \end{cases}$$

Part III: Applications

5 Spin Geometry on Complex Manifolds

5.1 Spin^c Structure Induced by an Almost Complex Structure

5.2 Dirac Operators on Complex Manifolds

6 Applications to Kähler Surfaces

Reference

[1] Andriy Haydys. Introduction to gauge theory. *arXiv preprint arXiv:1910.10436*, 2019.

[2] John Morgan. *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, volume 44. Princeton University Press, 1996.

[3] John Moore. *Lectures on Seiberg-Witten invariants*. Springer, 2009.

[4] Simon Kirwan Donaldson and Peter Kronheimer. *The geometry of four-manifolds*. Oxford university press, 1997.

[5] Robert Friedman and John Morgan. *Smooth four-manifolds and complex surfaces*, volume 27. Springer Science & Business Media, 2013.

[6] Blaine Lawson and Marie-Louise Michelsohn. *Spin Geometry (PMS-38), Volume 38*, volume 20. Princeton university press, 2016.

[7] John Milnor and James Stasheff. *Characteristic classes*. Number 76. Princeton university press, 1974.

[8] Glenys Luke and Alexander Mishchenko. *Vector bundles and their applications*, volume 447. Springer Science & Business Media, 2013.

[9] Jean-Pierre Schneiders. *Introduction to characteristic classes and index theory*. 2000.

[10] Edgar Brown Jr. The cohomology of $\mathrm{BSO}(n)$ and $\mathrm{BO}(n)$ with integer coefficients. In *Proc. Am. Math. Soc*, volume 85, pages 283–288, 1982.

[11] Ralph Cohen. Bundles, Homotopy, and Manifolds, 2023.

[12] Riccardo Benedetti and Paolo Lisca. Framing 3-manifolds with bare hands. *L'Enseignement Mathématique*, 64(3):395–413, 2019.

[13] Dietmar Salamon. *Spin Geometry and Sieberg-Witten Invariants*. ETH, 2000.

[14] Weiping Zhang. *Lectures on Chern-Weil theory and Witten deformations*, volume 4. World Scientific, 2001.

[15] Dusa McDuff, Dietmar Salamon, et al. *J-holomorphic curves and quantum cohomology*, volume 6. Citeseer, 1994.

[16] Jean-Louis Koszul. Sur certains groupes de transformations de lie. In *Colloque international du Centre National de la Recherche Scientifique*, volume 52, pages 137–141, 1953.